Signed graphs whose all Laplacian eigenvalues are main

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Main results

- For a graph G we consider the problem of the existence of a switching equivalent signed graph with Laplacian eigenvalues that are all main and the problem of determination of all switching equivalent signed graphs with this spectral property.
- Using a computer search we confirm that apart from K_2 every connected graph with at most 7 vertices switches to at least one signed graph with the required property. This fails to hold for exactly 22 connected graphs with 8 vertices.
- If G is a cograph without repeated eigenvalues, then we give an iterative solution and the complete solution in the particular case when G is a threshold graph.

What is a signed graph?

- Given a graph $G = (V(G), E(G))$, let $\sigma\colon E(G) \longrightarrow \{1, -1\}$. Then $\dot{G} = (G, \sigma)$ is a *signed graph* derived from its *underlying graph* G.
- The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the $(0, 1)$-adjacency matrix of the underlying graph G by reversing the sign of all 1s which correspond to negative edges.
- The *Laplacian matrix* of $\dot{G}$ is defined by $L_{\dot{G}} = D_{\dot{G}} - A_{\dot{G}}$, where $D_{\dot{G}}$ is the diagonal matrix of vertex degrees.
- The *eigenvalues* and the *spectrum* of $\dot{G}$ are identified to be the eigenvalues and the spectrum of $A_{\dot{G}}$, while the *Laplacian eigenvalues* and the *Laplacian spectrum* of $\dot{G}$ refer to the eigenvalues and the spectrum of $L_{\dot{G}}$.

Switching equivalent signed graphs

- We say that the signed graphs $\dot{G}$ and $\dot{H}$ are *switching equivalent* if there is a vertex subset $U \subseteq V(\dot{G})$ such that $\dot{H}$ is obtained by reversing the sign of every edge with one end in U and the other in $V(\dot{G}) \setminus U$.
- The underlying graphs of switching equivalent signed graphs are isomorphic. If the vertex labelling is transferred from the common underlying graph, then $\dot{G}$ is switching equivalent to $\dot{H}$ if and only if there is a diagonal matrix S of ± 1s, called the *switching matrix*, such that $A_{\dot{H}} = S^{-1} A_{\dot{G}} S$.
- In this case, we also have $L_{\dot{H}} = S^{-1} L_{\dot{G}} S$. Switching equivalence preserves the spectrum of $A_{\dot{G}}$ and the spectrum of $L_{\dot{G}}$.
- If $\mathbf{x}$ is an eigenvector of $A_{\dot{G}}$ (or $L_{\dot{G}}$), then $S\mathbf{x}$ is an eigenvector that corresponds to the same eigenvalue of $A_{\dot{H}}$ (or $L_{\dot{H}}$).

Main eigenvalues

- We say that an eigenvalue of $A_{\dot{G}}$ or $L_{\dot{G}}$ is *main* if the corresponding eigenspace contains an eigenvector that is non-orthogonal to the all-1 vector $\mathbf{j}$. For example, for every unsigned graph G, zero is the Laplacian eigenvalue associated with $\mathbf{j}$, and therefore zero is the main eigenvalue.
- It was proved that for every eigenvalue of a signed graph, there exists a switching equivalent signed graph in which this particular eigenvalue is main.
- Main eigenvalues are important for counting walks as well as for applications in control theory.

Motivation

- In Akbari et al. conjectured that, apart from K_2 and the graph obtained by deleting an edge of K_4, for every graph G there exists a switching equivalent signed graph $\dot{G}$ such that all the eigenvalues of $A_{\dot{G}}$ are main. In the same reference, the conjecture is confirmed for graphs with at most 9 vertices and also for Cayley graphs, distance-regular graphs, vertex-transitive and edge-transitive graphs, double stars and paths.
- We consider an analogue problem for Laplacian spectrum of signed graphs.

Problem

Problem

For a given graph G determine all switching equivalent signed graphs $\dot{G}$ such that all the eigenvalues of $A_{\dot{G}}$ (or $L_{\dot{G}}$) are main.

We use $\mathbf{s}$ to denote the vector equal to the main diagonal of the switching matrix S. We say that the pair $(A_G, \mathbf{s})$ (or $(L_G, \mathbf{s})$) is *mainable* if all the eigenvalues of $S^{-1}A_GS$ (or $S^{-1}L_GS$) are main.

Basic tools

Theorem 1

Let G and H be the graphs with Laplacian eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m = 0$ and $\mu_1, \mu_2, \dots, \mu_n = 0$, respectively. The Laplacian eigenvalues of $G \nabla H$ are $m + n$, $\lambda_1 + n, \lambda_2 + n, \dots, \lambda_{m-1} + n$, $\mu_1 + m, \mu_2 + m, \dots, \mu_{n-1} + m$ and 0.

If $\mathbf{x}$ is a Laplacian eigenvector of G orthogonal to $\mathbf{j}$ with a Laplacian eigenvalue λ, then its extension being zero on each vertex of H is a Laplacian eigenvector of $G \nabla H$ with eigenvalue $\lambda + n$, and similarly for the Laplacian eigenvectors of $G \nabla H$ that are formed on the basis of those of H. The Laplacian eigenvalue $m + n$ corresponds to the eigenvector with weight m on each vertex of G and $-n$ on each vertex of H.

Basic tools

Theorem 2

If G is a graph with n ($n \geq 6$) vertices, then

(i) G realizes $S_{1,n}$ if and only if
   (a) $G \cong (2K_1)\nabla(K_1 \cup H)$, for some graph H that realizes $S_{n-4,n-3}$ or
   (b) $G \cong K_1 \nabla H$, for some graph H that realizes $S_{n-1,n-1}$;

(ii) G realizes $S_{n-1,n}$ if and only if
   (a) $G \cong K_1\nabla(K_2 \cup H)$, for some graph H that realizes $S_{2,n-3}$ or
   (b) $G \cong K_1\nabla(K_1 \cup H)$, for some graph H that realizes $S_{n-2,n-2}$;

(iii) If $2 \leq i \leq n-2$, then G realizes $S_{i,n}$ if and only if $G \cong K_1\nabla(K_1 \cup H)$, for some graph H that realizes $S_{i-1,n-2}$.

$$S_{i,n} = \{0, 1, \dots, n\} \setminus \{i\}$$

Cographs & threshold graphs

- A *cograph* is a graph that does not contain the 4-vertex path P_4 as an induced subgraph.
- A *threshold graph* does not contain an induced subgraph isomorphic to the two copies of K_2, or the path P_4, or the cycle C_4 (we say that a threshold graph is $\{2K_2, P_4, C_4\}$-free).

- A cograph has the form $G \cup H$ or $G \nabla H$, where G, H are also cographs. It follows that the Laplacian spectrum of a cograph is integral.

Cographs without repeated eigenvalues I

Lemma 1

If $G \ncong K_1$ is a cograph without repeated eigenvalues, then either $G \cong K_2$ or G is constructed as in item (i.a), (ii.a) or (iii) of Theorem 2.

Lemma 2

Let G be a cograph that realizes $S_{1,n}$. Then $G \cong K_1$, $G \cong K_2$, $G \cong (2K_1)\nabla(K_1 \cup K_2)$, $G \cong (2K_1)\nabla(K_1 \cup P_3)$ or G is formed by taking a cograph that realizes $S_{1,n-8}$ and applying (iii), (ii.a) and (i.a) of Theorem 2, respectively.

Cographs without repeated eigenvalues II

Lemma 3

Let G be a cograph that realizes $S_{n-1,n}$. Then $G \cong K_2$, $G \cong P_3$, $G \cong K_1 \nabla (K_2 \cup P_3)$, $G \cong K_1 \nabla (K_2 \cup K_1 \nabla (K_1 \cup K_2))$ or G is formed by taking a cograph that realizes $S_{n-9,n-8}$ and applying (i.a), (iii) and (ii.a) of Theorem 2, respectively.

Lemma 4

Let G be a cograph that realizes $S_{i,n}$, for $2 \leq i \leq n-2$. Then G is formed by applying (iii) of Theorem 2 to a cograph that realizes $S_{i-1,n-2}$.

Cographs without repeated eigenvalues III

Lemma 5

There is a cograph G that realizes $S_{i,n}$ if and only if

(i) $i = 1$ with $n \equiv 1$ or $n \equiv 2 \pmod 4$ or $i = n - 1$ with $n \equiv 2$ or $n \equiv 3 \pmod 4$ or,

(ii) $2 \leq i \leq n - 2$ and either $n \equiv 2i - 1$ or $n \equiv 2i \pmod 4$.

Moreover, in each case, G is uniquely determined by the given spectrum.

Mainabiity of cographs with at least 9 vertices

Theorem 3

Let G be a cograph realizing $S_{1,n}$ which is obtained as in Lemma 2 from a cograph $H \ncong K_1$ that realizes $S_{1,n-8}$.
Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)^\intercal$ be a $(1,-1)$-vector and $\mathbf{s}' = (s_9, s_{10}, \ldots, s_n)^\intercal$ be the restriction of $\mathbf{s}$ (on H). If $(L_H, \mathbf{s}')$ is mainable, then $(L_G, \mathbf{s})$ is mainable if and only if $\langle \mathbf{s}, \mathbf{j} \rangle \neq 0$, $s_1 \neq s_2$, $s_5 \neq s_6$,
$-\langle \mathbf{s}', \mathbf{j_{n-8}} \rangle \notin \{s_7 + s_8, s_3 + s_4 + s_7 + s_8\}$.

Theorem 4

Let G be a cograph realizing $S_{n-1,n}$ which is obtained as in Lemma 3 from a cograph H that realizes $S_{n-9,n-8}$.
Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)^\intercal$ be a $(1,-1)$-vector and $\mathbf{s}' = (s_9, s_{10}, \ldots, s_n)^\intercal$ be the restriction of $\mathbf{s}$. If $(L_H, \mathbf{s}')$ is mainable, then $(L_G, \mathbf{s})$ is mainable if and only if $\langle \mathbf{s}, \mathbf{j} \rangle \neq 0$, $s_2 \neq s_3$, $s_6 \neq s_7$, $-\langle \mathbf{s}', \mathbf{j_{n-8}} \rangle \notin \{s_8, s_4 + s_5 + s_8\}$.

Mainability of cographs

Theorem 5

Let G be a cograph realizing $S_{i,n}$, for $2 \le i \le n-2$, which is obtained as in Lemma 4 from a cograph H that realizes $S_{i-1,n-2}$. A complete system of linearly independent eigenvectors of G is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -\frac{1}{n-1} \\ \mathbf{j} & \mathbf{y_1} & \mathbf{y_2} & \cdots & \mathbf{y_{n-3}} & -\frac{1}{n-2}\mathbf{j} & -\frac{1}{n-1}\mathbf{j} \end{bmatrix},$$

where $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_{n-3}}$ are linearly independent eigenvectors corresponding to non-zero eigenvalues of H.

Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)^\intercal$ be a $(1,-1)$-vector and $\mathbf{s}' = (s_3, s_4, \ldots, s_n)^\intercal$ be the restriction of $\mathbf{s}$. If $(L_H, \mathbf{s}')$ is mainable, then $(L_G, \mathbf{s})$ is mainable if and only if $\langle \mathbf{s}, \mathbf{j} \rangle \neq 0$.

Mainability of threshold graphs

Theorem 6

A graph G is a threshold graph without repeated eigenvalues if and only if G realizes $S_{i,n}$, where $n = 2i - 1$ or $n = 2i$ and $i \in \mathbb{N}$. Moreover, with an appropriate vertex labelling, linearly independent eigenvectors of G are given by the columns of

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -\frac{1}{n-1} \\ 1 & 0 & 0 & \cdots & 1 & -\frac{1}{n-2} & -\frac{1}{n-1} \\ 1 & 0 & 0 & \cdots & -\frac{1}{n-3} & -\frac{1}{n-2} & -\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & -\frac{1}{n-3} & -\frac{1}{n-2} & -\frac{1}{n-1} \\ 1 & 1 & -\frac{1}{2} & \cdots & -\frac{1}{n-3} & -\frac{1}{n-2} & -\frac{1}{n-1} \\ 1 & -1 & -\frac{1}{2} & \cdots & -\frac{1}{n-3} & -\frac{1}{n-2} & -\frac{1}{n-1} \end{bmatrix}. \tag{1}$$

If $\mathbf{s} = (s_1, s_2, \ldots, s_n)^\intercal$ is a $(1, -1)$-vector, then $(L_G, \mathbf{s})$ is mainable if and only if $\langle \mathbf{s}, \mathbf{j} \rangle \neq 0$ and $s_{n-1} \neq s_n$.

Additional results

Theorem 8

Let G and H be graphs of order n_1 and n_2 ($n_1, n_2 \geq 2$).

(i) For $\mathbf{s} = (s_1, s_2, \ldots, s_{n_2+1})^\intercal$, $\mathbf{s}' = (s_2, s_3, \ldots, s_{n_2+1})^\intercal$, if $(L_H, \mathbf{s}')$ is mainable, then $(L_{K_1 \nabla H}, \mathbf{s})$ is mainable if and only if $s_1 \neq -\langle \mathbf{s}', \mathbf{j_{n_2}} \rangle$.

(ii) For $\mathbf{s} = (s_1, s_2, \ldots, s_{n_1+n_2})^\intercal$, $\mathbf{s}' = (s_1, s_2, \ldots, s_{n_1})^\intercal$, $\mathbf{s}'' = (s_{n_1+1}, s_{n_1+2}, \ldots, s_{n_1+n_2})^\intercal$ if $(L_G, \mathbf{s}')$ and $(L_H, \mathbf{s}'')$ are mainable, then $(L_{G \nabla H}, \mathbf{s})$ is mainable if and only if $\langle \mathbf{s}, \mathbf{j} \rangle \neq 0$ and $\langle \mathbf{s}', \mathbf{j_{n_1}} \rangle \neq \frac{n_1}{n_2} \langle \mathbf{s}'', \mathbf{j_{n_2}} \rangle$.

- Applications in control theory. For a symmetric $n \times n$ matrix M and an $n \times 1$ vector $\mathbf{b}$, we say that a pair $(M, \mathbf{b})$ is *controllable* if every eigenvector of M is non-orthogonal to $\mathbf{b}$. It is known that if M has an eigenvalue of multiplicity at least two, then $(M, \mathbf{b})$ is not controllable for any choice of $\mathbf{b}$.
- There is a particular case when we say that a signed graph $\dot{G}$ is *controllable* if $A_{\dot{G}}$ has no eigenvector orthogonal to $\mathbf{j}$, and similarly it is called *Laplacian controllable* if the same holds for the Laplacian matrix $L_{\dot{G}}$ instead of $A_{\dot{G}}$. Equivalently, a signed graph is (Laplacian) controllable if all its (Laplacian) eigenvalues are simple and main.

Future work

- It can be of the interest to study mainabilty of some other classes of graphs.
- Mainability regarding the adjacency spectrum.

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Thank you!