Classes of strongly regular signed graphs and their relations with association schemes

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Signed graphs – basic facts I

- $G = (V(G), E(G))$ - a graph; $\sigma\colon E(G) \longrightarrow \{-1, +1\}$. Then $\dot{G} = (G, \sigma)$ is a *signed graph* derived from its *underlying graph* G and σ is a *sign function* or a *signature*.
- The edge set of a signed graph is composed of subsets of positive and negative edges. They induce the subgraphs denoted by $\dot{G}^+$ and $\dot{G}^-$, respectively. **Every graph can be interpreted as a signed graph without negative edges.**
- A signed graph is said to be *homogeneous* if all edges have the same sign.
- A signed graph is connected, regular or bipartite if the same holds for its underlying graph. A signed graph is *net-regular* if the difference between positive and negative edges incident with a vertex is a constant on the vertex set. This means that the all-ones vector $\mathbf{j}$ is an eigenvector of its adjacency matrix.

Signed graphs – basic facts II

- The $n \times n$ *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0,1)$-adjacency matrix of G by reversing the sign of all 1's which correspond to negative edges. The corresponding eigenvalues are real and form the spectrum of $\dot{G}$.
- A walk in a signed graph is *positive* if the number of its negative edges (counted with their multiplicity if there are repeated edges) is not odd. Otherwise, it is negative.
- A signed graph is said to be walk-regular if, for each vertex i, the difference between the numbers of positive and negative walks starting and terminating at i, and traversing k edges, is constant for each non-negative integer k,

Combinatorial definition

Definition [14]. We say that a signed graph $\dot{G}$ is *strongly regular* (for short, a *SRSG*) whenever it is regular and satisfies the following conditions:

(1) $\dot{G}$ is neither homogeneous complete or totally disconnected,

(2) there exists $a \in \mathbb{Z}$ such that $w_2(i,j) = a$, for all $i \overset{+}{\sim} j$,

(3) there exists $b \in \mathbb{Z}$ such that $w_2(i,j) = b$, for all $i \overset{-}{\sim} j$,

(4) there exists $c \in \mathbb{Z}$ such that $w_2(i,j) = c$, for all $i \nsim j$,

where $w_2(i,j)$ is the difference between the numbers of positive and negative walks traversing along 2 edges between the vertices i and j.

- The numbers (n, r, a, b, c) are called the *parameters of a SRSG*. It might occur that some of them are not unique, i.e. if $\dot{G}$ does not contain any positive edge then a is arbitrary, or if $\dot{G}$ is complete then c is arbitrary.
- If $\dot{G}$ is inhomogeneous, the parameters a, b are uniquely determined.
- If, in addition, $\dot{G}$ is non-complete then c is fixed, as well.

Definition in matrix form

- Definition written in the matrix form:

$$A_{\dot{G}}^2 = \frac{a}{2}(A_{\dot{G}} + A_G) - \frac{b}{2}(A_{\dot{G}} - A_G) + cA_{\overline{G}} + rI, \qquad (1)$$

  where r is the vertex degree of $\dot{G}$, and $\overline{G}$ is the complement of G.
- Important: For homogeneous SRSGs the identity (1) reduces to the well-known matrix identity defining strong regularity.

Symmetric association schemes – definition [3, 4]

- d be a positive integer and X a non-empty set.
- A symmetric *d-class association scheme* on X consists of a partition of $X \times X$ into the $d+1$ non-empty symmetric binary relations $R_0, R_1, \ldots, R_d$ satisfying the following conditions:
    - $R_0 = \{(x,x) \mid x \in X\}$,
    - if $(x,y) \in R_h$, then the number $z \in X$ such that $(x,z) \in R_i$ and $(y,z) \in R_j$ is a constant p_{ij}^h depending on i,j,h, but not on a choice of x,y.
- The numbers p_{ij}^h are called the *intersection numbers* of the scheme.

Symmetric association schemes – matrix form

- The ith binary relation R_i, also known as the ith associate class, can be represented by the matrix A_i in the following way:
  - A_i is the $(0, 1)$-matrix of order $|X|$ whose rows and columns are indexed by the elements of X,
  - $(A_i)_{xy} = 1$ if and only if $(x, y) \in R_i$,
  - $A_0 = I$, $A_i = A_i^T$, $\sum_{h=0}^{d} A_h = J$, $A_i A_j = \sum_{k=0}^{d} p_{ij}^{h} A_h$.
- Matrices A_i, $i \in \{0, 1, \ldots, d\}$ are also known as the *adjacency matrices* of the association scheme.
- The adjacency matrices generate a $(d + 1)$-dimensional commutative algebra $\mathcal{A}$ of symmetric matrices, which is called the Bose-Mesner algebra of the scheme.
- A $(d + 1)$-dimensional commutative algebra of adjacency matrices of a symmetric d-class association scheme is closed under ordinary matrix multiplication and under entrywise (Hadamard) multiplication.
- A vector space of symmetric matrices containing the identity matrix I and the all-ones matrix J that is closed under both ordinary and entrywise (Hadamard) multiplication is the Bose-Mesner algebra of an association scheme – Theorem 2.6.1 in [3]

A classification of SRSGs [7]

$\mathcal{C}_1$: SRSGs with $a = -b$, which are either complete or non-complete with $c \neq 0$. The identity (1) reduces to:

$$A_{\dot{G}}^2 + bA_{\dot{G}} = cA_{\overline{G}} + rI.$$

$\mathcal{C}_2$: SRSGs with $a = -b$, which are non-complete with $c = 0$. It holds:

$$A_{\dot{G}}^2 + bA_{\dot{G}} = rI.$$

$\mathcal{C}_3$: SRSGs with $a \neq -b$, which are either complete or non-complete with $c = \frac{a+b}{2}$. It holds:

$$A_{\dot{G}}^2 + \frac{b-a}{2}A_{\dot{G}} = \frac{a+b}{2}J + \left(r - \frac{a+b}{2}\right)I.$$

$\mathcal{C}_4$: SRSGs with $a \neq -b$, which are non-complete with $c = 0$. It holds:

$$A_{\dot{G}}^2 + \frac{b-a}{2}A_{\dot{G}} = \frac{a+b}{2}A_G + rI.$$

$\mathcal{C}_5$: SRSGs with $a \neq -b$, which are non-complete with $c \neq \frac{a+b}{2}$ and $c \neq 0$. It holds:

$$A_{\dot{G}}^2 + \frac{b-a}{2}A_{\dot{G}} = \frac{a+b-2c}{2}A_G + cJ + (r-c)I.$$

Class $\mathcal{C}_3$

- According to our knowledge SRSGs of $\mathcal{C}_3$ are studied only in [7]. They have properties similar to those of strongly regular graphs. We know the following:

Theorem (Koledin, Stanić [7])

If $\dot{G} \in \mathcal{C}_3$, then $\dot{G}$ is net-regular.

Theorem (Koledin, Stanić [7])

If $\dot{G} \in \mathcal{C}_3$, then $\dot{G}$ has exactly three eigenvalues, one of them being the net-degree with multiplicity one.

Theorem (Koledin, Stanić [7])

Let $\dot{G}$ be a connected inhomogeneous regular and net-regular signed graph with three eigenvalues. If its net-degree is a simple eigenvalue, then $\dot{G}$ is strongly regular.

- Also, if $\dot{G} \in \mathcal{C}_3$, then $\dot{G}$ is walk-regular [7].

Signed graphs from three class association schemes

- From a given symmetric 3-class association scheme, we can easily construct a SRSG in the following way: any of the relations R_1, R_2, R_3 of a scheme can be chosen to represent positive edges, any of the remaining two can be chosen to represent negative ones, and then the remaining relation represents non-edges.
- If we choose R_i to represent positive edges and R_j to represent negative ones, and if A_i and A_j are the the corresponding adjacency matrices of the scheme, then the adjacency matrix of $\dot{G}$ is $A_{\dot{G}} = A_i - A_j$.
- the parameters a, b, c of $\dot{G}$ can be expressed in terms of the intersection numbers in the following way:

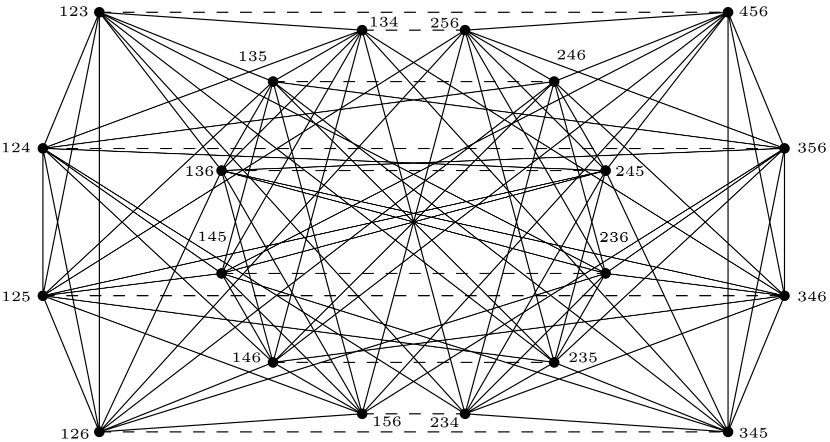
$$a = p^i_{ii} + p^i_{jj} - 2p^i_{ij}, \;\; b = p^j_{ii} + p^j_{jj} - 2p^j_{ij}, \;\; c = p^k_{ii} + p^k_{jj} - 2p^k_{ij}.$$

- SRSG constructed in this way is net-regular with net-degree $p^0_{ii} - p^0_{jj}$.

Johnson and Hamming signed graphs – definition [8]

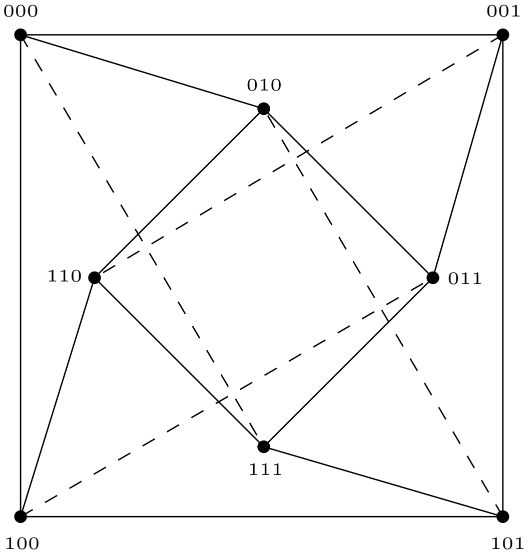
- The *d-class Johnson scheme* $J(v,d)$ is defined on the d-subsets of a v-set, with $d \leq \frac{v}{2}$. Two d-subsets are in relation R_i if and only if they intersect in $d-i$ elements.
- $J(v,d)$, $2 \leq d \leq \frac{v}{2}$ Johnson scheme; $A_1, A_2, \ldots, A_d$ its adjacency matrices. The *Johnson signed graph* $J(v,d)_{m,n}$, for $1 \leq m,n \leq d$ and $m \neq n$, is the signed graph determined by the adjacency matrix $A_{J(v,d)_{m,n}} = A_m - A_n$.
- The *d-class Hamming scheme* $H(d,q)$ has the set of all words of length d over an alphabet of q symbols as its vertex set. Two words are in relation R_i if and only if the Hamming distance between them is i (i.e., if and only if they differ in exactly i coordinates).
- $H(d,q)$, $d,q \geq 2$ Hamming scheme; $A_1, A_2, \ldots, A_d$ its adjacency matrices. The *Hamming signed graph* $H(d,q)_{m,n}$, for $1 \leq m,n \leq d$ and $m \neq n$, is the signed graph determined by the adjacency matrix $A_{H(d,q)_{m,n}} = A_m - A_n$.

Johnson signed graph $J(6,3)_{1,3}$



$J(6,3)_{1,3}$

Hamming signed graph $H(3,2)_{1,3}$



$H(3,2)_{1,3}$

Properties of Johnson and Hamming signed graphs

- Some properties of Johnson and Hamming signed graphs (expressions for their eigenvalues, the question of their strong regularity, etc.) can be found in [8].

Non-complete SRSGs of $\mathcal{C}_3$

- Are there non-complete SRSGs in the class $\mathcal{C}_3$? Yes!
- The three non-complete SRSGs in the class $\mathcal{C}_3$ we know about are constructed using symmetric association schemes (three class Johnson scemes, or four class Hamming scheme).
- Recall that there is a natural way to construct SRSGs using three class association schemes.
- Examples constructed via three class Johnson schemes: Johnson signed graphs $J(9,3)_{1,2}$ and $J(14,3)_{1,2}$.
- Third example: non-edges are obtained by merging two relations in a four-class Hamming association scheme – Hamming signed graph $H(4,4)_{1,2}$. This merging does not reduce the scheme to a three-class one.
- Questions:
    - Are there more non-complete SRSGs of $\mathcal{C}_3$ that can be constructed in a similar way (by merging classes in a d-class association scheme, $d \geq 4$)?
    - If $\dot{G} \in \mathcal{C}_3$ is complete then $A_{\dot{G}}$ is the Seidel matrix of a strongly regular graph, so adjacency matrices of $\dot{G}^+$ and $\dot{G}^-$ commute; the same holds for the three examples of non-complete SRSGs of $\mathcal{C}_3$;
    - Can we construct a SRSG $\dot{G}$ of $\mathcal{C}_3$, such that $\dot{G}^+$ and $\dot{G}^-$ are regular with non-commuting adjacency matrices?

SRSGs and their relation with association schemes I

- SRSGs with three eigenvalues:

Proposition

If a net-regular SRSG $\dot{G}$ with adjacency matrix $A_{\dot{G}}$ has three eigenvalues and its net-degree is not a simple one, then $\mathcal{A} = \langle A_{\dot{G}}, A_{\dot{G}}^2, I, J \rangle$ is Bose-Mesner algebra of a 3-class association scheme.

- The condition that net-degree is not a simple eigenvalue can not be omitted; the example is the already mentioned $H(4,4)_{1,2}$.
- The converse also does not hold, i.e. if for a SRSG $\dot{G}$ with adjacency matrix $A_{\dot{G}}$ and three eigenvalues the vector space $\mathcal{A} = \langle A_{\dot{G}}, A_{\dot{G}}^2, I, J \rangle$ is Bose-Mesner algebra of a 3-class association scheme we can not conclude that its net-degree is not a simple eigenvalue; for example SRSGs $J(9,3)_{1,2}$ and $J(14,3)_{1,2}$ obtained from three class association schemes

SRSGs and their relation with association schemes II

- Signed graphs with at most four eigenvalues:

Proposition

Let $\dot{G}$ be an inhomogeneous non-complete signed graph with adjacency matrix $A_{\dot{G}}$. Suppose that vector space $\mathcal{A} = \langle A_{\dot{G}}, A^2_{\dot{G}}, I, J \rangle$ is Bose-Mesner algebra of a 3-class association scheme, i.e. that $\mathcal{A}$ is closed under both ordinary and Hadamard multiplication. Then $\dot{G}$ is net-regular SRSG with at most four (distinct) eigenvalues.

- Again, we can not omit the assumption that $\dot{G}$ is neither homogeneous nor complete.

SRSGs and their relation with association schemes III

- SRSGs with four eigenvalues:

Proposition

If a net-regular SRSG $\dot{G}$ with adjacency matrix $A_{\dot{G}}$ has four eigenvalues and its net-degree is a simple one, then $\mathcal{A} = \langle A_{\dot{G}}, A^2_{\dot{G}}, I, J \rangle$ is Bose-Mesner algebra of a 3-class association scheme. Conversely, if an inhomogeneous non-complete signed graph $\dot{G}$ with adjacency matrix $A_{\dot{G}}$ has four eigenvalues and if $\mathcal{A} = \langle A_{\dot{G}}, A^2_{\dot{G}}, I, J \rangle$ is Bose-Mesner algebra of a 3-class association scheme, then $\dot{G}$ is net-regular SRSG, and its net-degree is a simple eigenvalue.

Walk-regularity I

- Walk-regularity of regular signed graphs:

Theorem (Anđelić, Koledin, Stanić [2])

A regular signed graph with at most three eigenvalues is walk-regular.

- Walk-regularity of net-regular SRSGs:

Proposition

The following statements hold:

(i) *Every net-regular SRSG with at most* 4 *eigenvalues is walk-regular;*

(ii) *Every net-regular SRSG with at most* 5 *eigenvalues and whose net-degree is a simple eigenvalue is walk-regular.*

- Walk-regularity of biparite SRSGs:

Proposition

Every bipartite SRSG is walk-regular.

Walk-regularity II

- Walk-regularity of SRSGs belonging to certain classes:

Proposition

Every SRSG of $\mathcal{C}_2 \cup \mathcal{C}_3$ is walk-regular. Let $\dot{G}$ be a SRSG with exactly 4 eigenvalues.

(i) *If $\dot{G} \in \mathcal{C}_1$ and the complement of its underlying graph is connected, then $\dot{G}$ is walk-regular.*

(ii) *If $\dot{G}$ is a connected signed graph of $\mathcal{C}_4$, then $\dot{G}$ is walk-regular.*

(iii) *If $\dot{G} \in \mathcal{C}_5$ and its parameters satisfy $(a+b-2c)(r-\lambda) \neq -2cn$, for every eigenvalue λ of the underlying graph G, then $\dot{G}$ is walk-regular.*

Walk-regularity III

- Walk-regularity of regular and net-regular signed graphs (but not necessarily strongly regular):

Proposition

The following statements hold:

(i) *Every regular and net-regular signed graph with at most 4 eigenvalues whose net-degree is a simple eigenvalue is walk-regular;*

(ii) *Every bipartite regular and net-regular signed graph with at most 6 eigenvalues whose net-degree is a simple eigenvalue is walk-regular.*

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That is all,

T H X!