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THE JACOBI-ORTHOGONALITY IN INDEFINITE SCALAR PRODUCT SPACES

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Abstract. We generalize the property of Jacobi-orthogonality to indefinite scalar product spaces. We compare various principles and investigate relations between Osserman, Jacobi-dual, and Jacobi-orthogonal algebraic curvature tensors. We show that every quasi-Clifford tensor is Jacobi-orthogonal. We prove that a Jacobi-diagonalizable Jacobi-orthogonal tensor is Jacobi-dual whenever $\mathcal{J}_X$ has no null eigenvectors for all nonnull X. We show that any algebraic curvature tensor of dimension 3 is Jacobi-orthogonal if and only if it is of constant sectional curvature. We prove that every 4-dimensional Jacobi-diagonalizable algebraic curvature tensor is Jacobi-orthogonal if and only if it is Osserman.

1. Introduction

Recently, Jacobi-orthogonal algebraic curvature tensors have been introduced as a new potential characterization of Riemannian Osserman tensors, and it has been proved that any Jacobi-orthogonal tensor is Osserman, while all known Osserman tensors are Jacobi-orthogonal [**3**]. We generalize the concept of Jacobi-orthogonality to indefinite scalar product spaces and investigate its relations with some important features such as Osserman, quasi-Clifford, and Jacobi-dual tensors.

Let $(\mathcal{V}, g)$ be a scalar product space of dimension n, that is, $\mathcal{V}$ is an n-dimensional vector space over $\mathbb{R}$, while g is a nondegenerate symmetric bilinear form on $\mathcal{V}$. The sign of the squared norm, $\varepsilon_X = g(X, X)$, distinguishes all vectors $X \in \mathcal{V} \setminus \{0\}$ into three different types. A vector $X \in \mathcal{V}$ is spacelike if $\varepsilon_X > 0$; timelike if $\varepsilon_X < 0$; null if $\varepsilon_X = 0$ and $X \neq 0$. Especially, a vector $X \in \mathcal{V}$ is nonnull if $\varepsilon_X \neq 0$ and it is unit if $\varepsilon_X \in \{-1, 1\}$. We say that X and Y are mutually orthogonal and write $X \perp Y$ if $g(X, Y) = 0$. For $X \perp Y$ we have

$$\varepsilon_{\alpha X+\beta Y} = g(\alpha X + \beta Y, \alpha X + \beta Y) = \alpha^2 \varepsilon_X + \beta^2 \varepsilon_Y. \tag{1.1}$$

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An important relation between null, timelike, and spacelike vectors is given in the following lemma (see [**1**, Lemma 1]).

Lemma 1.1. *Every null N from a scalar product space $\mathcal{V}$ can be decomposed as $N = S + T$, where $S, T \in \mathcal{V}$, $S \perp T$, and $\varepsilon_S = -\varepsilon_T$.*

We say that a subspace W of an indefinite scalar product space $(\mathcal{V}, g)$ is totally isotropic if it consists only of null vectors, which implies that any two vectors from W are mutually orthogonal. In what follows we will use the following well-known statement about an isotropic supplement of W (see [**2**, Proposition 1]).

Lemma 1.2. *If $\mathcal{W} \leqslant \mathcal{V}$ is a totally isotropic subspace with a basis $(N_1, \dots, N_k)$, then there exists a totally isotropic subspace $\mathcal{U} \leqslant \mathcal{V}$, disjoint from $\mathcal{W}$, with a basis $(M_1, \dots, M_k)$, such that $g(N_i, M_j) = \delta_{ij}$ holds for $1 \leqslant i, j \leqslant k$.*

A quadri-linear map $R\colon \mathcal{V}^4 \to \mathbb{R}$ is said to be an algebraic curvature tensor on $(\mathcal{V}, g)$ if it satisfies the usual $\mathbb{Z}_2$ symmetries as well as the first Bianchi identity. More concretely, an algebraic curvature tensor $R \in \mathfrak{T}^0_4(\mathcal{V})$ has the properties

$$R(X,Y,Z,W) = -R(Y,X,Z,W), \tag{1.2}$$

$$R(X,Y,Z,W) = -R(X,Y,W,Z), \tag{1.3}$$

$$R(X,Y,Z,W) = R(Z,W,X,Y), \tag{1.4}$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0, \tag{1.5}$$

for all $X, Y, Z, W \in \mathcal{V}$.

The basic example of an algebraic curvature tensor is the tensor R^1 of constant sectional curvature 1, defined by

$$R^1(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W).$$

Furthermore, skew-adjoint endomorphisms J on $\mathcal{V}$ generate new examples by

$$R^J(X,Y,Z,W) = g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W) + 2g(JX,Y)g(JZ,W).$$

A quasi-Clifford family of rank m is an anti-commutative family of skew-adjoint endomorphisms J_i, for $1 \leqslant i \leqslant m$, such that $J_i^2 = c_i \operatorname{Id}$, for $c_i \in \mathbb{R}$. In other words, a quasi-Clifford family satisfies the Hurwitz-like relations, $J_iJ_j + J_jJ_i = 2\delta_{ij}c_i \operatorname{Id}$, for $1 \leqslant i, j \leqslant m$. We say that an algebraic curvature tensor R is quasi-Clifford if

$$R = \mu_0 R^1 + \sum_{i=1}^{m} \mu_i R^{J_i}, \tag{1.6}$$

for some $\mu_0, \dots, \mu_m \in \mathbb{R}$, where J_i, for $1 \leqslant i \leqslant m$, is some associated quasi-Clifford family. Especially, R is Clifford if it is quasi-Clifford with $c_i = -1$ for all $1 \leqslant i \leqslant m$. Let us remark that Clifford tensors were observed in [**10**, **12**] and quasi-Clifford tensors were considered in [**2**].

If $E_1, E_2, \dots, E_n \in \mathcal{V}$ are mutually orthogonal units, we say that $(E_1, \dots, E_n)$ is an orthonormal basis of $\mathcal{V}$. The signature of a scalar product space $(\mathcal{V}, g)$ is an ordered pair (p, q), where p is the number of negative ε_{E_i}, while q is the number of positive ε_{E_i}. We say that R is Riemannian if $p = 0$; Lorentzian if $p = 1$; Kleinian if $p = q$.

Raising the index we obtain the algebraic curvature operator $\mathcal{R} = R^\sharp \in \mathfrak{T}_3^1(\mathcal{V})$. The polarized Jacobi operator is the linear map $\mathcal{J}\colon \mathcal{V}^3 \to \mathcal{V}$ defined by

$$\mathcal{J}(X,Y)Z = \frac{1}{2}(\mathcal{R}(Z,X)Y + \mathcal{R}(Z,Y)X)$$

for all $X,Y,Z \in \mathcal{V}$. For each $X \in \mathcal{V}$ the Jacobi operator $\mathcal{J}_X$ is a linear self-adjoint operator $\mathcal{J}_X\colon \mathcal{V} \to \mathcal{V}$ defined by $\mathcal{J}_X Y = \mathcal{J}(X,X)Y = \mathcal{R}(Y,X)X$ for all $Y \in \mathcal{V}$. Using the three-linearity of $\mathcal{R}$, for every $Z \in \mathcal{V}$ we get

$$\mathcal{J}_{tX}Z = \mathcal{R}(Z,tX)(tX) = t^2\mathcal{R}(Z,X)X = t^2\mathcal{J}_X Z, \tag{1.7}$$

$$\mathcal{J}_{X+Y}Z = \mathcal{R}(Z,X+Y)(X+Y) = \mathcal{J}_X Z + 2\mathcal{J}(X,Y)Z + \mathcal{J}_Y Z. \tag{1.8}$$

Using (1.4) we get that any two Jacobi operators satisfy the compatibility condition, which means that $g(\mathcal{J}_X Y, Y) = g(\mathcal{J}_Y X, X)$ holds for all $X,Y \in \mathcal{V}$. Since $\mathcal{J}_X X = 0$ and $g(\mathcal{J}_X Y, X) = 0$, we conclude that for any nonnull $X \in \mathcal{V}$ the Jacobi operator $\mathcal{J}_X$ is completely determined by its restriction $\widetilde{\mathcal{J}}_X\colon X^\perp \to X^\perp$ called the reduced Jacobi operator.

Let R be an algebraic curvature tensor and $\widetilde{\omega}_X(\lambda) = \det(\lambda\,\mathrm{Id} - \widetilde{\mathcal{J}}_X)$. We say that R is timelike Osserman if $\widetilde{\omega}_X$ is independent of unit timelike $X \in \mathcal{V}$. We say that R is spacelike Osserman if $\widetilde{\omega}_X$ is independent of unit spacelike $X \in \mathcal{V}$. Naturally, R is called Osserman if it is both timelike and spacelike Osserman. It is known that timelike Osserman and spacelike Osserman conditions are equivalent (see [**9**]). It is easy to see that every quasi-Clifford tensor is Osserman (see [**2**]).

We say that R is k-stein if there exist constants $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$\mathrm{tr}((\mathcal{J}_X)^j) = (\varepsilon_X)^j c_j \tag{1.9}$$

holds for each $1 \leqslant j \leqslant k$ and all $X \in \mathcal{V}$. It is known that an algebraic curvature tensor of dimension n is Osserman if and only if it is n-stein (see [**11**, Lemma 1.7.3]).

We say that R is Jacobi-diagonalizable if $\mathcal{J}_X$ is diagonalizable for any nonnull X. In this case we have

$$\mathcal{V} = \mathrm{Span}\{X\} \oplus \bigoplus_{l=1}^{k} \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\lambda_l\,\mathrm{Id}), \tag{1.10}$$

where $\varepsilon_X\lambda_1, \ldots, \varepsilon_X\lambda_k$ are all eigenvalues of $\widetilde{\mathcal{J}}_X$ and $\oplus$ denotes the direct orthogonal sum.

The duality principle in the Riemannian setting (g is positive definite) appeared in [**14**]. Its generalization to a pseudo-Riemannian setting (see [**4**, **5**]) is given by the implication

$$\mathcal{J}_X Y = \varepsilon_X\lambda Y \implies \mathcal{J}_Y X = \varepsilon_Y\lambda X. \tag{1.11}$$

If (1.11) holds for all mutually orthogonal unit $X,Y \in \mathcal{V}$, then we say that R is weak Jacobi-dual, and if (1.11) holds for all $X,Y \in \mathcal{V}$ with the restriction $\varepsilon_X \neq 0$, we say that R is Jacobi-dual. If R is Jacobi-diagonalizable, it is sufficient to prove that it is weak Jacobi-dual which we see in the following lemma (see [**1**, **4**]).

Lemma 1.3. *Every Jacobi-diagonalizable algebraic curvature tensor is Jacobi-dual if and only if it is weak Jacobi-dual.*

The condition that R is Jacobi-diagonalizable is strong enough to provide the equivalence between Osserman and Jacobi-dual property in a pseudo-Riemannian setting.

Theorem 1.1. [**13**] *Every Jacobi-diagonalizable algebraic curvature tensor is Osserman if and only if it is Jacobi-dual.*

2. The Jacobi-orthogonality

In [**3**] we introduced a new concept of Jacobi-orthogonality, and here we generalize it to a pseudo-Riemannian setting. We say that an algebraic curvature tensor is Jacobi-orthogonal if the implication

$$X \perp Y \implies \mathcal{J}_X Y \perp \mathcal{J}_Y X \tag{2.1}$$

holds for all unit $X, Y \in \mathcal{V}$. However, it is easy to extend this for all $X, Y \in \mathcal{V}$, which we see in the following lemma.

Lemma 2.1. *If an algebraic curvature tensor is Jacobi-orthogonal, then* (2.1) *holds for all* $X, Y \in \mathcal{V}$.

Proof. Suppose R is Jacobi-orthogonal and $X \perp Y$. The assertion is obvious for $X = 0$ or $Y = 0$. If X and Y are both nonnull, (2.1) holds after we rescale them using (1.7).

We consider the case $\varepsilon_X \neq 0$ and $\varepsilon_Y = 0$. Since $X^\perp$ is nondegenerate and contains null Y, according to Lemma 1.1, there exist $S, T \in X^\perp$ such that $Y = S + T$, $S \perp T$, $\varepsilon_S = -\varepsilon_T > 0$. Since X, S, T are nonnull, $X \perp S$, and $X \perp T$, using (2.1) we get $g(\mathcal{J}_X S, \mathcal{J}_S X) = 0$ and $g(\mathcal{J}_X T, \mathcal{J}_T X) = 0$. Hence, using (1.8) and denoting $K = \mathcal{J}_S X$, $L = 2\mathcal{J}(S,T)X$, $M = \mathcal{J}_T X$, $P = \mathcal{J}_X S$, and $Q = \mathcal{J}_X T$, we calculate

$$\begin{aligned} g(\mathcal{J}_X(S+\lambda T), \mathcal{J}_{S+\lambda T} X) &= g(P + \lambda Q, K + \lambda L + \lambda^2 M) \\ &= (g(P,M) + g(Q,L))\lambda^2 + (g(Q,K) + g(P,L))\lambda. \end{aligned} \tag{2.2}$$

For every $\lambda \neq \pm 1$, using (1.1) we get $\varepsilon_{S+\lambda T} = \varepsilon_S(1-\lambda^2) \neq 0$, so $X \perp S + \lambda T$ implies $g(\mathcal{J}_X(S+\lambda T), \mathcal{J}_{S+\lambda T} X) = 0$, where (2.2) gives $g(P,M) + g(Q,L) = 0$ and $g(Q,K) + g(P,L) = 0$. Hence, (2.2) for $\lambda = 1$ implies $g(\mathcal{J}_X(S+T), \mathcal{J}_{S+T} X) = 0$ which proves (2.1) for one nonnull and one null vector.

It remains to prove (2.1) for two null vectors $X = N_1$ and $Y = N_2$. If they are linearly dependent, we have $N_1 = \xi N_2$ for some $\xi \in \mathbb{R}$, so $\mathcal{J}_{N_1} N_2 = 0$ and therefore (2.1) holds. If N_1 and N_2 are linearly independent mutually orthogonal vectors, then they form a basis (N_1, N_2) of the totally isotropic subspace $\operatorname{Span}\{N_1, N_2\} \leqslant \mathcal{V}$. According to Lemma 1.2 there exists a basis (M_1, M_2) of a totally isotropic subspace of $\mathcal{V}$ that is disjoint from $\operatorname{Span}\{N_1, N_2\}$ and $g(N_i, M_j) = \delta_{ij}$, for $1 \leqslant i, j \leqslant 2$. We can decompose $N_2 = S + T$, where $S = (N_2 + M_2)/2$, $T = (N_2 - M_2)/2$, and $S, T \in N_1^\perp$. Since $\varepsilon_S = -\varepsilon_T = 1/2$ and $S \perp T$, repeating the same procedure as in the previous part of the proof, we get (2.2) and using already proved implication (2.1) for nonnull $S+\lambda T$ and null vector N_1 we have (2.1) for null vectors $X = N_1$ and $Y = N_2$. $\square$

Sometimes, it is useful to add the tensor of constant sectional curvature to the observed algebraic curvature tensor R.

LEMMA 2.2. *If an algebraic curvature tensor R is Jacobi-orthogonal, then for each $\mu \in \mathbb{R}$, the tensor $R + \mu R^1$ is Jacobi-orthogonal.*

PROOF. Let $\mathcal{J}'$ be the Jacobi operator associated with the algebraic curvature tensor $R' = R + \mu R^1$, while X and Y are mutually orthogonal unit vectors. Using $\mathcal{J}_X Y \perp X$, $\mathcal{J}_Y X \perp Y$, and the Jacobi-orthogonality of R, we get

$$g(\mathcal{J}'_X Y, \mathcal{J}'_Y X) = g(\mathcal{J}_X Y + \mu\varepsilon_X Y, \mathcal{J}_Y X + \mu\varepsilon_Y X) = g(\mathcal{J}_X Y, \mathcal{J}_Y X) = 0,$$

which means that $R' = R + \mu R^1$ is Jacobi-orthogonal. $\square$

In the Riemannian setting we know that every Clifford algebraic curvature tensor is Jacobi-orthogonal (see [**3**]). We use Lemma 2.2 to give a generalization to a pseudo-Riemannian setting.

THEOREM 2.1. *Every quasi-Clifford algebraic curvature tensor is Jacobi-orthogonal.*

PROOF. Let $J_1, J_2, \dots, J_m$ be a quasi-Clifford family associated to a quasi-Clifford algebraic curvature tensor of the form (1.6). Consider $R = \sum_{i=1}^m \mu_i R^{J_i}$ and units $X \perp Y$. Since the endomorphism J_i is skew-adjoint, we have $g(J_i X, X) = 0$, which yields

$$\begin{aligned}
\mathcal{J}_X Y &= \sum_{i=1}^m \mu_i \mathcal{R}^{J_i}(Y, X)X \\
&= \sum_{i=1}^m \mu_i (g(J_i Y, X)J_i X - g(J_i X, X)J_i Y + 2g(J_i Y, X)J_i X) \\
&= 3\sum_{i=1}^m \mu_i g(J_i Y, X)J_i X,
\end{aligned}$$

and similarly $\mathcal{J}_Y X = 3\sum_{j=1}^m \mu_j g(J_j X, Y)J_j Y$. For units $X \perp Y$, using that J_i is skew-adjoint for $i \in \{1, 2, \dots, m\}$ and the Hurwitz-like relations, we get

$$\begin{aligned}
g(\mathcal{J}_X Y, \mathcal{J}_Y X) &= g\Big(3\sum_{i=1}^m \mu_i g(J_i Y, X)J_i X,\ 3\sum_{j=1}^m \mu_j g(J_j X, Y)J_j Y\Big) \\
&= 9\sum_{i,j} \mu_i\mu_j g(J_i Y, X)g(J_j X, Y)g(J_i X, J_j Y) \\
&= 9\sum_{i,j} \mu_i\mu_j g(X, J_i Y)g(X, J_j Y)g(X, J_i J_j Y)
\end{aligned}$$

$$= \frac{9}{2} \sum_{i,j} \mu_i \mu_j g(X, J_i Y) g(X, J_j Y) g(X, (J_i J_j + J_j J_i) Y)$$
$$= \frac{9}{2} \sum_{i,j} 2\delta_{ij} c_i \mu_i \mu_j g(X, J_i Y) g(X, J_j Y) g(X, Y) = 0,$$

which proves that R is Jacobi-orthogonal. According to Lemma 2.2 it follows that the quasi-Clifford $R + \mu_0 R^1$ is Jacobi-orthogonal. $\square$

In order to examine the Jacobi-duality of a Jacobi-diagonalizable Jacobi-orthogonal algebraic curvature tensor, we give the following two lemmas which give us information about $\mathcal{J}_Y X$, where Y is an eigenvector of $\mathcal{J}_X$ for a nonnull vector $X \in \mathcal{V}$.

Lemma 2.3. *Let R be a Jacobi-diagonalizable Jacobi-orthogonal algebraic curvature tensor. If $X \in \mathcal{V}$ is a nonnull vector and $Y \in \mathcal{V}_i(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda_i \,\mathrm{Id})$, then $\mathcal{J}_Y X \in \mathrm{Span}\{X\} \oplus \mathcal{V}_i(X)$.*

Proof. If $\widetilde{\mathcal{J}}_X$ has only one eigenvalue $\varepsilon_X \lambda_i$, then $\mathrm{Span}\{X\} \oplus \mathcal{V}_i(X) = \mathcal{V}$, so the statement is obvious. Let $Z \in \mathcal{V}_j(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda_j \,\mathrm{Id})$ for $\lambda_j \neq \lambda_i$ and $L = Y + tZ$, where $t \in \mathbb{R}$. Since $Y \in \mathcal{V}_i(X) \leqslant X^\perp$ and $Z \in \mathcal{V}_j(X) \leqslant X^\perp$ we have $L \perp X$, so using the Jacobi-orthogonality of R, Lemma 2.1, and (1.3), we get

$$\begin{aligned} 0 = g(\mathcal{J}_L X, \mathcal{J}_X L) &= g(\mathcal{R}(X, Y + tZ)(Y + tZ), \mathcal{J}_X Y + t\mathcal{J}_X Z) \\ &= R(X, Y + tZ, Y + tZ, \varepsilon_X \lambda_i Y + t\varepsilon_X \lambda_j Z) \\ &= \varepsilon_X (t\lambda_j - t\lambda_i) R(X, Y + tZ, Y, Z) \\ &= \varepsilon_X (\lambda_i - \lambda_j) R(X, Z, Z, Y) t^2 + \varepsilon_X (\lambda_j - \lambda_i) R(X, Y, Y, Z) t. \end{aligned}$$

Since this holds for all $t \in \mathbb{R}$, we conclude that the coefficient of t is zero and because of $\varepsilon_X(\lambda_j - \lambda_i) \neq 0$ we obtain $R(X, Y, Y, Z) = 0$, and therefore $\mathcal{J}_Y X \perp Z$, which holds for every $Z \in \mathcal{V}_j(X)$, whenever $\lambda_j \neq \lambda_i$. Since R is Jacobi diagonalizable, we have (1.10), where $\varepsilon_X \lambda_1, \dots, \varepsilon_X \lambda_k$ are all (different) eigenvalues of $\widetilde{\mathcal{J}}_X$, so we conclude that $\mathcal{J}_Y X \in \mathrm{Span}\{X\} \oplus \mathcal{V}_i(X)$. $\square$

Lemma 2.4. *Let R be a Jacobi-diagonalizable Jacobi-orthogonal algebraic curvature tensor. If $X \in \mathcal{V}$ is a nonnull vector and $Y \in \mathcal{V}(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda \,\mathrm{Id})$, then $\mathcal{J}_Y X = \varepsilon_Y \lambda X + Z$, where $\varepsilon_Z = 0$.*

Proof. Let $\mathcal{J}_Y X = \alpha X + Z$, where $Z \in X^\perp$ and $\alpha \in \mathbb{R}$. The compatibility of Jacobi operators gives $g(\mathcal{J}_Y X, X) = g(\mathcal{J}_X Y, Y)$, so $g(\alpha X + Z, X) = g(\varepsilon_X \lambda Y, Y)$. Hence, $\alpha \varepsilon_X = \lambda \varepsilon_X \varepsilon_Y$ and since $\varepsilon_X \neq 0$, we get $\alpha = \varepsilon_Y \lambda$ and $\mathcal{J}_Y X = \varepsilon_Y \lambda X + Z$. From $Y \in \mathcal{V}(X) \leqslant X^\perp$, we get $g(\varepsilon_X Y - t\varepsilon_Y X, X + tY) = 0$, so using that R is Jacobi-orthogonal, Lemma 2.1, (1.7), (1.8), and the equalities $2\mathcal{J}(X, Y)Y = -\mathcal{J}_Y X$, $2\mathcal{J}(X, Y)X = -\mathcal{J}_X Y$, we obtain

$$\begin{aligned} 0 &= g\big(\mathcal{J}_{X+tY}(\varepsilon_X Y - t\varepsilon_Y X), \mathcal{J}_{\varepsilon_X Y - t\varepsilon_Y X}(X + tY)\big) \\ &= g\big(\varepsilon_X \mathcal{J}_X Y - t\varepsilon_X \mathcal{J}_Y X + t^2 \varepsilon_Y \mathcal{J}_X Y - t^3 \varepsilon_Y \mathcal{J}_Y X, \\ &\qquad \varepsilon_X^2 \mathcal{J}_Y X + t\varepsilon_X \varepsilon_Y \mathcal{J}_X Y + t^2 \varepsilon_X \varepsilon_Y \mathcal{J}_Y X + t^3 \varepsilon_Y^2 \mathcal{J}_X Y\big). \end{aligned}$$

Since every $t \in \mathbb{R}$ is a root of the polynomial equation

$$g(\mathcal{J}_{X+tY}(\varepsilon_X Y - t\varepsilon_Y X), \mathcal{J}_{\varepsilon_X Y - t\varepsilon_Y X}(X + tY)) = 0,$$

we conclude that all coefficients are zero, and therefore the coefficient of t is $\varepsilon_X^2 \varepsilon_Y g(\mathcal{J}_X Y, \mathcal{J}_X Y) - \varepsilon_X^3 g(\mathcal{J}_Y X, \mathcal{J}_Y X) = 0$, which implies $\varepsilon_Y \varepsilon_{\mathcal{J}_X Y} = \varepsilon_X \varepsilon_{\mathcal{J}_Y X}$ because $\varepsilon_X \neq 0$, and therefore $\varepsilon_Y \varepsilon_{\varepsilon_X \lambda Y} = \varepsilon_X \varepsilon_{\varepsilon_Y \lambda X + Z}$. Since $Z \in X^\perp$, using (1.1), we get $\varepsilon_Y \varepsilon_X^2 \lambda^2 \varepsilon_Y = \varepsilon_X(\varepsilon_Y^2 \lambda^2 \varepsilon_X + \varepsilon_Z)$, which gives $\varepsilon_Z = 0$. □

As a consequence of the last two lemmas, we easily get the following theorem.

THEOREM 2.2. *Every Jacobi-diagonalizable Jacobi-orthogonal algebraic curvature tensor is Jacobi-dual, when $\mathcal{J}_X$ has no null eigenvectors for all nonnull X.*

PROOF. Let X and Y be two mutually orthogonal vectors such that $\varepsilon_X \neq 0$ and $\mathcal{J}_X Y = \varepsilon_X \lambda Y$. Using Lemma 2.4 we get $\mathcal{J}_Y X = \varepsilon_Y \lambda X + Z$, where $\varepsilon_Z = 0$, while Lemma 2.3 gives $Z \in \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda \operatorname{Id})$. If Z is null, then it is not an eigenvector of $\mathcal{J}_X$, which implies $Z = 0$, so $\mathcal{J}_Y X = \varepsilon_Y \lambda X$, which proves that R is Jacobi-dual. □

3. Low dimensional cases

In this section we consider the cases of small dimension $n \in \{3, 4\}$. In dimension 3 we obtain the following expected result.

THEOREM 3.1. *Every algebraic curvature tensor of dimension* 3 *is Jacobi-orthogonal if and only if it is of constant sectional curvature.*

PROOF. Suppose R is a 3-dimensional algebraic curvature tensor of constant sectional curvature μ. Since the zero tensor is Jacobi-orthogonal, Lemma 2.2 implies that $R = 0 + \mu R^1$ is Jacobi-orthogonal.

Conversely, suppose R is a Jacobi-orthogonal algebraic curvature tensor of dimension 3. Let (E_1, E_2, E_3) be an arbitrary orthonormal basis of $\mathcal{V}$, $\varepsilon_i = \varepsilon_{E_i}$, for $1 \leqslant i \leqslant 3$, and $R_{ijkl} = R(E_i, E_j, E_k, E_l)$, for $i, j, k, l \in \{1, 2, 3\}$. Using the formula $\mathcal{R}(E_i, E_j)E_k = \sum_l \varepsilon_l R_{ijkl} E_l$ and (1.3), we obtain $\mathcal{J}_{E_1} E_2 = \varepsilon_2 R_{2112} E_2 + \varepsilon_3 R_{2113} E_3$ and $\mathcal{J}_{E_2} E_1 = \varepsilon_1 R_{1221} E_1 + \varepsilon_3 R_{1223} E_3$. Hence, since $E_1 \perp E_2$ and R is Jacobi-orthogonal, we get $R_{2113} R_{1223} = 0$. Using rescaling we obtain

$$R(B, A, A, C) R(A, B, B, C) = 0, \tag{3.1}$$

for an arbitrary orthogonal basis (A, B, C) which consists of nonnull vectors.

Consider the basis $X = E_1$, $Y = E_2 + tE_3$, $Z = t\varepsilon_3 E_2 - \varepsilon_2 E_3$, where $t > 1$. Using (1.1), we get $\varepsilon_X = \varepsilon_1 \neq 0$, $\varepsilon_Y = \varepsilon_2 + t^2 \varepsilon_3 \neq 0$, $\varepsilon_Z = t^2 \varepsilon_3^2 \varepsilon_2 + \varepsilon_2^2 \varepsilon_3 \neq 0$, $g(X, Y) = 0$, $g(X, Z) = 0$ and $g(Y, Z) = t\varepsilon_3 \varepsilon_2 - t\varepsilon_2 \varepsilon_3 = 0$, so (X, Y, Z) is an orthogonal basis which consists of nonnull vectors, so applying (3.1) we get

$$\begin{aligned} 0 &= R(E_2 + tE_3, E_1, E_1, t\varepsilon_3 E_2 - \varepsilon_2 E_3) R(E_1, E_2 + tE_3, E_2 + tE_3, t\varepsilon_3 E_2 - \varepsilon_2 E_3) \\ &= (-\varepsilon_2 R_{2113} + (\varepsilon_3 R_{2112} - \varepsilon_2 R_{3113})t + \varepsilon_3 R_{3112} t^2)(R_{1223} + tR_{1323})(-\varepsilon_2 - \varepsilon_3 t^2). \end{aligned}$$

Since this holds for every $t > 1$, we conclude that the coefficient of t in the polynomial is 0. Thus, using (1.3) and $\varepsilon_2 \neq 0$, we get

$$\varepsilon_2 R_{2113} R_{1332} + (\varepsilon_3 R_{2112} - \varepsilon_2 R_{3113}) R_{1223} = 0,$$

so (3.1) for $(A,B,C)=(E_3,E_1,E_2)$ implies $(\varepsilon_3 R_{2112}-\varepsilon_2 R_{3113})R_{1223}=0$. Rescaling the vectors we obtain

$$(\varepsilon_C R(B,A,A,B)-\varepsilon_B R(C,A,A,C))R(A,B,B,C)=0, \tag{3.2}$$

for an arbitrary orthogonal basis (A,B,C) which consists of nonnull vectors.

Let (E_1,E_2,E_3) be an arbitrary orthonormal basis of $\mathcal{V}$ and (p,q,r) a permutation of the set $\{1,2,3\}$. Let $s_1=R_{2113}$, $s_2=R_{1223}$, $s_3=R_{1332}$, $k_1=\varepsilon_2\varepsilon_3 R_{3223}$, $k_2=\varepsilon_1\varepsilon_3 R_{3113}$, and $k_3=\varepsilon_1\varepsilon_2 R_{2112}$. From (3.1) we get for $(A,B,C)=(E_p,E_q,E_r)$ gives $s_p s_q=0$, and since this holds for an arbitrary permutation (p,q,r) of the set $\{1,2,3\}$, we get that at least two of s_1, s_2, s_3 are zero. Let $s_p=s_q=0$ and suppose $s_r\neq 0$. Hence, (3.2) for $(A,B,C)=(E_q,E_r,E_p)$ multiplied by $\varepsilon_p\varepsilon_q\varepsilon_r\neq 0$, gives $(k_p-k_r)s_r=0$, which implies $k_p=k_r$.

Consider $A=E_1+tE_3$, $B=E_2$, $C=\varepsilon_3 tE_1-\varepsilon_1 E_3$, for $t>1$. Using (1.1) we get $\varepsilon_A=\varepsilon_1+t^2\varepsilon_3\neq 0$, $\varepsilon_B=\varepsilon_2\neq 0$, $\varepsilon_C=\varepsilon_3^2 t^2\varepsilon_1+\varepsilon_1^2\varepsilon_3=t^2\varepsilon_1+\varepsilon_3\neq 0$, $g(A,B)=0$, $g(A,C)=\varepsilon_3 t\varepsilon_1-t\varepsilon_1\varepsilon_3=0$, and $g(B,C)=0$, so $(E_1+tE_3,E_2,\varepsilon_3 tE_1-\varepsilon_1 E_3)$ is an orthogonal basis which consists of nonnull vectors and applying (3.2), (1.1), (1.2), (1.3), (1.4) we compute

$$\begin{aligned}\big((t^2\varepsilon_1+\varepsilon_3)(R_{2112}+2R_{1223}t+R_{3223}t^2)-\varepsilon_2 R_{3113}(\varepsilon_1+\varepsilon_3 t^2)^2\big)&\\ \times\big(-\varepsilon_1 R_{1223}+(\varepsilon_3 R_{2112}-\varepsilon_1 R_{3223})t+\varepsilon_3 R_{1223}t^2\big)&=0.\end{aligned}$$

This holds for every $t>1$, so the coefficient of t is zero, and using $\varepsilon_1^2\varepsilon_2^2\varepsilon_3^2=1$, we obtain

$$-2\varepsilon_1\varepsilon_3 R_{1223}^2+(\varepsilon_1\varepsilon_2 R_{2112}-\varepsilon_1\varepsilon_3 R_{3113})(\varepsilon_1\varepsilon_2 R_{2112}-\varepsilon_2\varepsilon_3 R_{3223})=0.$$

Hence, $-2\varepsilon_1\varepsilon_3 s_2^2+(k_3-k_2)(k_3-k_1)=0$. Thus, using the basis (E_q,E_r,E_p) instead of (E_1,E_2,E_3), we get

$$-2\varepsilon_q\varepsilon_p s_r^2+(k_p-k_r)(k_p-k_q)=0, \tag{3.3}$$

which with $k_p=k_r$ and $\varepsilon_q\varepsilon_p\neq 0$ gives $s_r=0$, which contradicts $s_r\neq 0$. Thus, $s_p=s_q=s_r=0$, which implies

$$R_{2113}=R_{1223}=R_{1332}=0.$$

Hence, (3.3) gives $(k_p-k_r)(k_p-k_q)=0$ for any permutation (p,q,r) of the set $\{1,2,3\}$, so at least two of differences k_3-k_2, k_3-k_1, and k_2-k_1 are zero, which implies $k_1=k_2=k_3=\mu$, and therefore

$$R_{2112}=\varepsilon_1\varepsilon_2\mu,\quad R_{3113}=\varepsilon_1\varepsilon_3\mu,\quad R_{3223}=\varepsilon_2\varepsilon_3\mu.$$

Since an algebraic curvature tensor of dimension 3 is uniquely determined by its 6 components of tensor: R_{2113}, R_{1223}, R_{1332}, R_{2112}, R_{3113}, R_{3223} (see [**15**, pp. 142–144]), the previous equalities imply that R is of constant sectional curvature μ. $\square$

Since every 3-dimensional R is 1-stein if and only if it is of constant sectional curvature (see [**6**, Proposition 1.120]), the previous theorem implies that every 3-dimensional R is Jacobi-orthogonal if and only if it is Osserman. In the following theorem we prove a similar result in dimension 4 using an additional hypothesis that R is Jacobi-diagonalizable.

THEOREM 3.2. *Every Jacobi-diagonalizable algebraic curvature tensor of dimension* 4 *is Osserman if and only if it is Jacobi-orthogonal.*

PROOF. Suppose R is a Jacobi-diagonalizable Osserman algebraic curvature tensor of dimension 4. It is well-known that a Lorentzian Osserman algebraic curvature tensor has constant sectional curvature (see [**7**, **8**]), so it is of the form $R = \mu R^1$. Hence, using that 0 is Jacobi-orthogonal and applying Lemma 2.2, we conclude that Lorentzian R is Jacobi-orthogonal. It remains to deal with a Riemannian or Kleinian R. Let X and Y be mutually orthogonal unit vectors in $\mathcal{V}$. Denote $X = E_1$. Since R is Jacobi-diagonalizable, there exists an orthonormal eigenbasis (E_1, E_2, E_3, E_4) related to $\mathcal{J}_{E_1}$ such that $\mathcal{J}_{E_1}E_i = \varepsilon_1\lambda_i E_i$, for $2 \leqslant i \leqslant 4$, where $\varepsilon_j = \varepsilon_{E_j}$, for $1 \leqslant j \leqslant 4$. Since R is not Lorentzian, we have $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, as well as $\varepsilon_i^2 = 1$, for $1 \leqslant i \leqslant 4$. Denoting $R_{ijkl} = R(E_i, E_j, E_k, E_l)$, we get $R_{i11j} = g(\mathcal{J}_{E_1}E_i, E_j) = g(\varepsilon_1\lambda_i E_i, E_j) = \varepsilon_1\lambda_i\delta_{ij}\varepsilon_i$. Hence,

$$R_{2112} = \varepsilon_1\varepsilon_2\lambda_2, \quad R_{3113} = \varepsilon_1\varepsilon_3\lambda_3, \quad R_{4114} = \varepsilon_1\varepsilon_4\lambda_4, \tag{3.4}$$

$$R_{2113} = R_{2114} = R_{3114} = 0. \tag{3.5}$$

According to Theorem 1.1, a Jacobi-diagonalizable Osserman R is Jacobi-dual. Thus, $\mathcal{J}_{E_1}E_i = \varepsilon_1\lambda_i E_i$, for $2 \leqslant i \leqslant 4$, implies $\mathcal{J}_{E_i}E_1 = \varepsilon_i\lambda_i E_1$, so $\mathcal{J}_{E_i}E_1 \perp E_j$ for $2 \leqslant j \leqslant 4$, which means $0 = g(\mathcal{J}_{E_i}E_1, E_j) = R_{1iij}$ and therefore

$$R_{1223} = R_{1224} = R_{1332} = R_{1334} = R_{1442} = R_{1443} = 0. \tag{3.6}$$

Since R is 1-stein, (1.9) holds for $j = 1$ and we get $\sum_i \varepsilon_i\varepsilon_x R_{ixxi} = c_1$, for $x \in \{1, 2, 3, 4\}$ (see [**1**]). Thus, using (1.4) we obtain

$$\begin{aligned}
\varepsilon_1\varepsilon_2 R_{2112} + \varepsilon_1\varepsilon_3 R_{3113} + \varepsilon_1\varepsilon_4 R_{4114} &= c_1,\\
\varepsilon_1\varepsilon_2 R_{2112} + \varepsilon_2\varepsilon_3 R_{3223} + \varepsilon_2\varepsilon_4 R_{4224} &= c_1,\\
\varepsilon_1\varepsilon_3 R_{3113} + \varepsilon_2\varepsilon_3 R_{3223} + \varepsilon_3\varepsilon_4 R_{4334} &= c_1,\\
\varepsilon_1\varepsilon_4 R_{4114} + \varepsilon_2\varepsilon_4 R_{4224} + \varepsilon_3\varepsilon_4 R_{4334} &= c_1.
\end{aligned}$$

Therefore, subtracting the sum of the two of these equations from the sum of the remaining two equations, we get $\varepsilon_2\varepsilon_3 R_{3223} = \varepsilon_1\varepsilon_4 R_{4114}$, $\varepsilon_2\varepsilon_4 R_{4224} = \varepsilon_1\varepsilon_3 R_{3113}$, and $\varepsilon_3\varepsilon_4 R_{4334} = \varepsilon_1\varepsilon_2 R_{2112}$. Using (3.4), we obtain

$$R_{3223} = \varepsilon_1\varepsilon_4\lambda_4, \quad R_{4224} = \varepsilon_1\varepsilon_3\lambda_3, \quad R_{4334} = \varepsilon_1\varepsilon_2\lambda_2. \tag{3.7}$$

For a 1-stein R we also have additional equalities $\sum_i \varepsilon_i R_{ixyi} = 0$ for $1 \leqslant x \neq y \leqslant 4$ (see [**1**]). Using them for $(x, y) \in \{(2, 3), (2, 4), (3, 4)\}$, (1.2), (1.3), and (1.4), we conclude $R_{2443} = -\varepsilon_1\varepsilon_4 R_{2113}$, $R_{2334} = -\varepsilon_1\varepsilon_3 R_{2114}$, and $R_{3224} = -\varepsilon_1\varepsilon_2 R_{3114}$. Thus, using (3.5), we obtain

$$R_{2443} = R_{2334} = R_{3224} = 0. \tag{3.8}$$

Since Osserman R is 2-stein, (1.9) holds for $j = 2$, so we get $\operatorname{tr}(\mathcal{J}_{E_1})^2 = (\varepsilon_{E_1})^2 c_2$, which gives

$$\lambda_2^2 + \lambda_3^2 + \lambda_4^2 = c_2. \tag{3.9}$$

Since R is 2-stein, for all $1 \leqslant x \neq y \leqslant 4$, we get known 2-stein equations (see [**1**])

$$2 \sum_{1 \leqslant i,j \leqslant 4} \varepsilon_i \varepsilon_j R_{ixxj} R_{iyyj} + \sum_{1 \leqslant i,j \leqslant 4} \varepsilon_i \varepsilon_j (R_{ixyj} + R_{iyxj})^2 = 2\varepsilon_x \varepsilon_y c_2.$$

For $(x,y) = (2,3)$, using (1.2), (1.3), (1.4), (3.5), (3.6), and (3.8), we get

$$\begin{aligned} &2\varepsilon_1^2 R_{2112} R_{3113} + 2\varepsilon_4^2 R_{4224} R_{4334} + \varepsilon_1 \varepsilon_4 (R_{1234} + R_{1324})^2 \\ &+ \varepsilon_2 \varepsilon_3 (-R_{3223})^2 + \varepsilon_3 \varepsilon_2 (-R_{3223})^2 + \varepsilon_4 \varepsilon_1 (R_{4231} + R_{4321})^2 = 2\varepsilon_2 \varepsilon_3 c_2. \end{aligned}$$

Using (3.7), we compute $4\varepsilon_2\varepsilon_3\lambda_2\lambda_3 + 2\varepsilon_2\varepsilon_3(R_{1234} + R_{1324})^2 + 2\varepsilon_2\varepsilon_3\lambda_4^2 = 2\varepsilon_2\varepsilon_3 c_2$. Since $2\varepsilon_2\varepsilon_3 \neq 0$, we get $c_2 - \lambda_4^2 - 2\lambda_2\lambda_3 = (R_{1234} + R_{1324})^2$ and using (3.9) we get $(\lambda_3 - \lambda_2)^2 = (R_{1234} + R_{1324})^2$. Similarly, using (1.2), (1.3), (1.4), and (1.5) we obtain

$$\begin{aligned} (\lambda_2 - \lambda_4)^2 &= (R_{1243} + R_{1423})^2 = (R_{1324} - 2R_{1234})^2, \\ (\lambda_4 - \lambda_3)^2 &= (R_{1432} + R_{1342})^2 = (R_{1234} - 2R_{1324})^2. \end{aligned}$$

Hence, we get

$$\begin{aligned} &s_4(\lambda_3 - \lambda_2) = R_{1234} + R_{1324}, \; s_3(\lambda_2 - \lambda_4) = R_{1324} - 2R_{1234}, \\ &s_2(\lambda_4 - \lambda_3) = R_{1234} - 2R_{1324}, \end{aligned} \tag{3.10}$$

where $s_2, s_3, s_4 \in \{-1, 1\}$. According to the pigeonhole principle, at least two of s_2, s_3, s_4 are the same. First, suppose $s_i = s_j = -s_k$, where (i,j,k) is a permutation of $(2,3,4)$. Summing the equations in (3.10) we obtain

$$(s_3 - s_4)\lambda_2 + (s_4 - s_2)\lambda_3 + (s_2 - s_3)\lambda_4 = 0,$$

and we conclude $(s_j - s_k)\lambda_i + (s_k - s_i)\lambda_j = 0$, so $\lambda_i = \lambda_j$. Notice that substituting s_k by $-s_k$ does not change (3.10) and provides $s_2 = s_3 = s_4$.

If $s_2 = s_3 = s_4 = -1$, then substituting eigenvectors E_2, E_3 and E_4 with $-E_2$, $-E_3$ and $-E_4$, respectively, we conclude that R_{1234} and R_{1324} change the sign, as well as s_2, s_3, s_4. Therefore, without loss of generality we can suppose $s_2 = s_3 = s_4 = 1$, and get

$$\begin{aligned} R_{1234} - 2R_{1324} &= \lambda_4 - \lambda_3, \\ R_{1324} - 2R_{1234} &= \lambda_2 - \lambda_4, \\ R_{1234} + R_{1324} &= \lambda_3 - \lambda_2. \end{aligned} \tag{3.11}$$

For an arbitrary $Y \perp X = E_1$ there exist real numbers k_2, k_3, k_4 such that $Y = k_2E_2 + k_3E_3 + k_4E_4$, and therefore

$$\mathcal{J}_X Y = \mathcal{J}_{E_1}(k_2E_2 + k_3E_3 + k_4E_4) = k_2\varepsilon_1\lambda_2E_2 + k_3\varepsilon_1\lambda_3E_3 + k_4\varepsilon_1\lambda_4E_4.$$

Using (1.2)–(1.5), (3.5), (3.6), (3.11), and $\mathcal{R}(X,Y)Z = \sum_i \varepsilon_i R(X,Y,Z,E_i)E_i$ we calculate

$$\begin{aligned} \mathcal{J}_Y X = \mathcal{J}_{k_2E_2+k_3E_3+k_4E_4} E_1 &= \mathcal{R}(E_1, k_2E_2 + k_3E_3 + k_4E_4)(k_2E_2 + k_3E_3 + k_4E_4) \\ &= k_2^2\varepsilon_1 R_{2112}E_1 + k_2k_3\varepsilon_4 R_{1234}E_4 + k_2k_4\varepsilon_3 R_{1243}E_3 \\ &\quad + k_3k_2\varepsilon_4 R_{1324}E_4 + k_3^2\varepsilon_1 R_{1331}E_1 + k_3k_4\varepsilon_2 R_{1342}E_2 \\ &\quad + k_4k_2\varepsilon_3 R_{1423}E_3 + k_4k_3\varepsilon_2 R_{1432}E_2 + k_4^2\varepsilon_1 R_{1441}E_1 \end{aligned}$$

$$
\begin{aligned}
&= (k_2^2\varepsilon_1 R_{2112} + k_3^2\varepsilon_1 R_{1331} + k_4^2\varepsilon_1 R_{1441})E_1 + k_3k_4\varepsilon_2(R_{1342} + R_{1432})E_2 \\
&\quad + k_2k_4\varepsilon_3(R_{1243} + R_{1423})E_3 + k_2k_3\varepsilon_4(R_{1234} + R_{1324})E_4 \\
&= (k_2^2\varepsilon_2\lambda_2 + k_3^2\varepsilon_3\lambda_3 + k_4^2\varepsilon_4\lambda_4)E_1 + k_3k_4\varepsilon_2(R_{1234} - 2R_{1324})E_2 \\
&\quad + k_2k_4\varepsilon_3(R_{1324} - 2R_{1234})E_3 + k_2k_3\varepsilon_4(R_{1234} + R_{1324})E_4 \\
&= (k_2^2\varepsilon_2\lambda_2 + k_3^2\varepsilon_3\lambda_3 + k_4^2\varepsilon_4\lambda_4)E_1 + k_3k_4\varepsilon_2(\lambda_4 - \lambda_3)E_2 \\
&\quad + k_2k_4\varepsilon_3(\lambda_2 - \lambda_4)E_3 + k_2k_3\varepsilon_4(\lambda_3 - \lambda_2)E_4.
\end{aligned}
$$

Thus, using that (E_1, E_2, E_3, E_4) is an orthonormal basis, we compute

$$
\begin{aligned}
g(\mathcal{J}_X Y, \mathcal{J}_Y X) &= k_2k_3k_4\varepsilon_1\varepsilon_2\lambda_2(\lambda_4 - \lambda_3)g(E_2, E_2) \\
&\quad + k_2k_3k_4\varepsilon_1\varepsilon_3\lambda_3(\lambda_2 - \lambda_4)g(E_3, E_3) + k_2k_3k_4\varepsilon_1\varepsilon_4\lambda_4(\lambda_3 - \lambda_2)g(E_4, E_4) \\
&= \varepsilon_1 k_2k_3k_4(\lambda_2(\lambda_4 - \lambda_3) + \lambda_3(\lambda_2 - \lambda_4) + \lambda_4(\lambda_3 - \lambda_2)) = 0,
\end{aligned}
$$

which proves that R is Jacobi-orthogonal.

Conversely, let R be a Jacobi-diagonalizable Jacobi-orthogonal algebraic curvature tensor of dimension 4. First, we prove that R is weak Jacobi-dual. Let X and Y be mutually orthogonal unit vectors in $\mathcal{V}$ such that $\mathcal{J}_X Y = \varepsilon_X \lambda Y$. Our aim is to prove $\mathcal{J}_Y X = \varepsilon_Y \lambda X$. Since R is Jacobi-diagonalizable and Jacobi-orthogonal, X is nonnull and $Y \in \mathcal{V}(X) = \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda \operatorname{Id})$, using Lemma 2.3 and Lemma 2.4, we get $\mathcal{J}_Y X = \varepsilon_Y \lambda X + Z$, where $\varepsilon_Z = 0$ and $Z \in \mathcal{V}(X) \leqslant X^\perp$. Moreover, since $g(Z, Y) = g(\mathcal{J}_Y X - \varepsilon_Y \lambda X, Y) = g(X, \mathcal{J}_Y Y) - \varepsilon_Y \lambda g(X, Y) = 0$, it follows that $Z \perp Y$, so we conclude $Z \in \operatorname{Span}\{X, Y\}^\perp$.

We discuss two cases. The case where $\operatorname{Span}\{X, Y\}^\perp$ is a definite subspace of $\mathcal{V}$ is easy since $\varepsilon_Z = 0$ and $Z \in \operatorname{Span}\{X, Y\}^\perp$ imply $Z = 0$.

It remains to deal with the case where $\operatorname{Span}\{X, Y\}^\perp$ is indefinite ($\varepsilon_X = \varepsilon_Y$ for a Lorentzian R, $\varepsilon_X = -\varepsilon_Y$ for a Kleinian R, while for a Riemannian R there is no such case). Since $\mathcal{J}_Y X = \varepsilon_Y \lambda X + Z$, our aim is to prove $Z = 0$. We assume $Z \neq 0$, where $\varepsilon_Z = 0$ implies Z is null. Since R is Jacobi-diagonalizable, we know $\mathcal{V}(X)$ is nondegenerate such as $\operatorname{Span}\{Y\}^\perp \cap \mathcal{V}(X)$ which contains null vector Z, so its dimension is at least 2. Thus, since $Y \in \mathcal{V}(X) \leqslant X^\perp$, we get $\dim \mathcal{V}(X) = 3$. Therefore, $\mathcal{V}(X) = X^\perp$ and $\widetilde{\mathcal{J}}_X = \varepsilon_X \lambda \operatorname{Id}$. There exists $W \in \operatorname{Span}\{X, Y\}^\perp$ such that $\varepsilon_W = -\varepsilon_Y$ and we write $Y = (Y - tW) + tW$ for $t > 1$. Since $Y - tW$, $tW \in \mathcal{V}(X)$, we have $\mathcal{J}_X(Y - tW) = \varepsilon_X \lambda (Y - tW)$ and $\mathcal{J}_X(tW) = \varepsilon_X \lambda tW$. Using $W \perp Y$ and (1.1), we get $\varepsilon_{Y-tW} = \varepsilon_Y + t^2\varepsilon_W = (1 - t^2)\varepsilon_Y$ and $\varepsilon_{tW} = t^2\varepsilon_W$. Therefore $\operatorname{sgn}(\varepsilon_{Y-tW}) = \operatorname{sgn}(\varepsilon_{tW}) = -\operatorname{sgn}(\varepsilon_Y)$ and we apply the solved case to X, $Y - tW$ and X, tW to obtain $\mathcal{J}_{Y-tW} X = \varepsilon_{Y-tW} \lambda X$ and $\mathcal{J}_{tW} X = \varepsilon_{tW} \lambda X$. Using (1.8) and $\mathcal{J}(tW, tW)X = \mathcal{J}_{tW} X$, we compute

$$
\begin{aligned}
\mathcal{J}_Y X = \mathcal{J}_{(Y-tW)+tW} X &= \mathcal{J}_{Y-tW} X + 2\mathcal{J}(Y - tW, tW)X + \mathcal{J}_{tW} X \\
&= \varepsilon_{Y-tW} \lambda X + 2t\mathcal{J}(Y, W)X - 2\mathcal{J}_{tW} X + \mathcal{J}_{tW} X \\
&= \varepsilon_{Y-tW} \lambda X + 2t\mathcal{J}(Y, W)X - \varepsilon_{tW} \lambda X = \varepsilon_Y \lambda X + 2t\mathcal{J}(Y, W)X.
\end{aligned}
$$

Since $\mathcal{J}_Y X = \varepsilon_Y \lambda X + 2t\mathcal{J}(Y, W)X$ holds for all $t > 1$, we get $2\mathcal{J}(Y, W)X = 0$ and $\mathcal{J}_Y X = \varepsilon_Y \lambda X$, contrary to assumption that $Z \neq 0$, so $Z = 0$.

Therefore, R is weak Jacobi-dual and since R is Jacobi-diagonalizable, using Lemma 1.3, we conclude that R is Jacobi-dual. Finally, Theorem 1.1 implies that R is Osserman. $\square$

Especially, since Riemannian curvature tensors are Jacobi-diagonalizable, we get that every algebraic curvature tensor on a positive definite scalar product space of dimension 4 is Osserman if and only if it is Jacobi-orthogonal.

At the end, we conclude that the Jacobi-orthogonal property is very important and useful in characterizing Osserman tensors in pseudo-Riemannian settings.

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