

Perfect Matching Complexes of Honeycomb Graphs

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Submitted: Sep 13, 2022; Accepted: Apr 30, 2023; Published: Jun 16, 2023

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Abstract

The *perfect matching complex* of a graph is the simplicial complex on the edge set of the graph with facets corresponding to perfect matchings of the graph. This paper studies the perfect matching complexes, $\mathcal{M}_p(H_{k \times m \times n})$, of honeycomb graphs. For $k = 1$, $\mathcal{M}_p(H_{1 \times m \times n})$ is contractible unless $n \geq m = 2$, in which case it is homotopy equivalent to the $(n - 1)$ -sphere. Also, $\mathcal{M}_p(H_{2 \times 2 \times 2})$ is homotopy equivalent to the wedge of two 3-spheres. The proofs use discrete Morse theory.

Mathematics Subject Classifications: 05C70, 05E45, 55P15, 57M15

*This article is based on work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors participated in the 2020/2021 Summer Research in Mathematics Program of the Mathematical Sciences Research Institute, Berkeley, California. M. Jelić Milutinović has been supported by the Project No. 7744592 MEGIC “Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics” of the Science Fund of Serbia, and by the Faculty of Mathematics University of Belgrade through the grant (No. 451-03-68/2022-14/200104) by the Ministry of Education, Science, and Technological Development of the Republic of Serbia.

⁰2020 Mathematics Subject Classification: 05C70 (05E45, 55P15, 57M15)

Keywords: perfect matching, simplicial complex, honeycomb graph, homotopy type, plane partition

1 Introduction and Background

There has been a great deal of research on the topology of simplicial complexes associated with graphs. Much of the early work in this area concerned matching complexes of complete graphs and of complete bipartite graphs, called “the matching complex” and “chessboard complexes,” respectively. This was often motivated by connections with other areas of algebra, topology and combinatorics. For example, the topology of the matching complex of the complete graph was used to study the partially ordered set of p -subgroups of a finite group [24]. It also proved useful in the study of the posets of partitions with restricted block sizes [25]. And [29] used matching complexes of complete bipartite graphs to address a computational geometry problem: counting the hyperplanes that bisect sets of points in general position. For an extensive discussion of the history and applications of matching complexes, see the survey [28] by Wachs, and references therein.

Other complexes associated with graphs have also been studied since the 1990s (e.g., [16]), but much activity was initiated by the dissertation and subsequent book of Jonsson [14]. In particular, there has been much work on independence complexes of graphs (for example, [1, 2, 9, 10, 22]). The more recent work on matching complexes of various types of graphs includes line tilings [4, 21], grid graphs [7, 12, 20], and more [3, 13, 18, 23, 27].

One particular type of graph is the honeycomb graph (a planar graph that is a tiling of hexagons), studied, for example, in [13, 21]. In his exploration of the topology of matching complexes [14], Jonsson suggests honeycomb graphs to be of interest for further study. In addition, perfect matchings in honeycomb graphs have been of interest in chemistry [15], and, recently, in “topological quantum chemistry” [6]. Furthermore, as we shall see, honeycomb graphs have a close connection with plane partitions [17]. In this paper we focus on honeycomb graphs and on a subcomplex of the matching complex, generated by the faces corresponding to perfect matchings.

1.1 Perfect matching complexes

A *matching* of a simple graph G is a set of edges of G , no two of which share a vertex. The *matching complex* of G , denoted by $\mathcal{M}(G)$, is the simplicial complex whose vertex set is the set of edges of G , and whose facets (maximal faces) correspond to maximal matchings of G .

Definition 1. Let G be a simple graph. A *perfect matching* of G is a matching that covers all vertices of the graph. The *perfect matching complex* of G , denoted by $\mathcal{M}_p(G)$, is the subcomplex of the matching complex whose facets correspond to perfect matchings of G .

Thus, if $\sigma = \{e_1, \dots, e_k\}$ is an arbitrary subset of the edges $E(G)$, σ is a simplex (face) in the complex $\mathcal{M}_p(G)$ if and only if there is a perfect matching P of G such that $\sigma \subseteq P$.

If the graph G admits at least one perfect matching, the perfect matching complex $\mathcal{M}_p(G)$ is a full-dimensional subcomplex of the matching complex $\mathcal{M}(G)$. Otherwise, if there is no perfect matching of the graph G , we consider the perfect matching complex to be void. Note that even finding the number of perfect matchings of a graph is not easy.

The number of perfect matchings of a graph can be expressed in terms of permanents of associated matrices [19]. It is known that computing the permanent is $\#P$ -complete [26].

It turns out that the perfect matching complexes of complete graphs, complete bipartite graphs, paths and cycles are either their entire matching complexes, or complexes that can easily be determined. Whenever G has an odd number of vertices, the perfect matching complex is void, so we focus only on graphs with even number of vertices.

- *Complete graph* $G = K_{2n}$. Every matching is a subset of at least one perfect matching, so $\mathcal{M}_p(K_{2n}) = \mathcal{M}(K_{2n})$.
- *Complete bipartite graph* $G = K_{m,n}$. If $m \neq n$, G does not contain a perfect matching, so $\mathcal{M}_p(K_{m,n})$ is the void complex. When $m = n$, every matching is a subset of a perfect matching, and $\mathcal{M}_p(K_{n,n}) = \mathcal{M}(K_{n,n})$.
- *Path* $G = P_{2n}$. If we denote the vertices of the path by a_1, a_2, \dots, a_{2n} respectively, there is only one perfect matching on G , containing edges $\{a_{2i-1}, a_{2i}\}$, $i \in \{1, \dots, n\}$. Therefore $\mathcal{M}_p(P_{2n})$ is a simplex Δ^{n-1} on n vertices.
- *Cycle* $G = C_{2n}$. There are exactly two perfect matchings on C_{2n} , and they are disjoint, so the complex $\mathcal{M}_p(C_{2n})$ is the disjoint union of two simplices of dimension $n - 1$, and $\mathcal{M}_p(C_{2n})$ is homotopy equivalent to the 0-sphere S^0 .

1.2 Bijection between perfect matchings of honeycombs and plane partitions

A **honeycomb graph** $H = H_{k \times m \times n}$ is a hexagonal tiling whose congruent, opposite sides are of length k, m and n hexagons.

In 2019, discrete Morse theory was used to determine the connectedness bounds of d -dimensional faces of matching complexes of $1 \times 1 \times n$ and $2 \times 1 \times n$ honeycomb graphs for $n \geq 1$ [13]. It was later shown by Matsushita that the homotopy type for matching complexes of $1 \times 2 \times n$ honeycomb graphs is a wedge of spheres [21]. Beyond the line of hexagons, the homotopy types of matching complexes of honeycomb graphs have been quite elusive. In this paper, we will consider the perfect matching complex of a honeycomb graph and prove that this subcomplex for $H_{1 \times m \times n}$ is contractible or homotopy equivalent to a sphere using the bijection between perfect matchings on honeycomb graphs and plane partitions. We also find the homotopy type of the matching complex of one larger honeycomb graph: $H_{2 \times 2 \times 2}$.

A **plane partition** is a two dimensional array of integers that are non-increasing moving from left to right and top to bottom. We define a plane partition $P_{k \times m \times n}$ through a $k \times m$ matrix whose entries are less than or equal to n and follow the non-increasing conditions. A plane partition can be visualized as a pile of unit cubes in the positive octant of \mathbb{R}^3 following the non-increasing conditions. The perfect matchings of a honeycomb graph are in bijection with the rhombus tilings of a hexagonal region of equilateral triangles, which are in bijection with plane partitions. See, for example, [17], which gives the number of plane partitions (and hence of perfect matchings) of honeycomb graphs under various symmetry groups. For an example see Figure 4 on page 8. We will use this

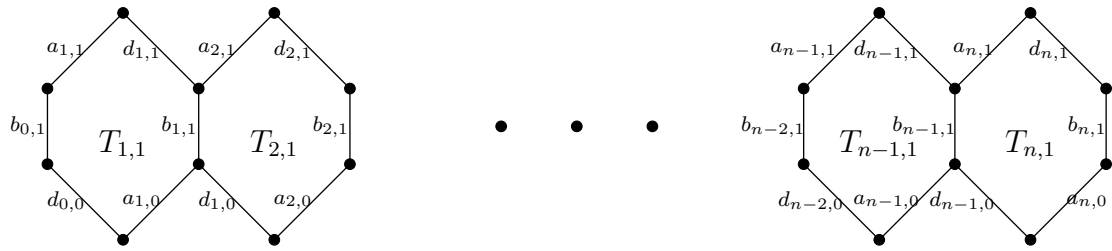


Figure 1: Labeled line of n hexagons

well-known bijection to determine the homotopy type of the perfect matching complexes of $H_{1 \times m \times n}$ and $H_{2 \times 2 \times 2}$.

2 Hexagonal line tiling

In this section we use the nerve theorem, an important theorem in topology, to find the homotopy type of the perfect matching complex of a line of hexagons. For an exposition in the combinatorial context of simplicial complexes, see [5].

Definition 2. The *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $\mathcal{N}(A_i)$ with vertex set I and $\sigma \subseteq I$ a face of $\mathcal{N}(A_i)$ if and only if $\bigcap_{i \in \sigma} A_i \neq \emptyset$.

Theorem 3. Suppose Δ is a simplicial complex, and $(\Delta_i)_{i \in I}$ is a family of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$. If every nonempty finite intersection $\bigcap_{i \in J} \Delta_i$ ($J \subseteq I$) is contractible, then Δ and the nerve $\mathcal{N}(\Delta_i)$ are homotopy equivalent.

Theorem 4. Let $H_n = H_{1 \times 1 \times n}$ be the graph of a line of n hexagons. Let $\mathcal{M}_p(H_n)$ be the perfect matching complex of H_n . For $n \geq 2$, $\mathcal{M}_p(H_n)$ is contractible.

Proof. Let $n \geq 2$. Label the graph H_n as in Figure 1.

For each vertical segment $b_{i,1}$ ($0 \leq i \leq n$), there is a unique perfect matching for H_n containing $b_{i,1}$. This is the matching

$$\{a_{1,1}, d_{0,0}, a_{2,1}, d_{1,0}, \dots, a_{i,1}, d_{i-1,0}, b_{i,1}, d_{i+1,1}, a_{i+1,0}, \dots, d_{n-1,1}, a_{n-1,0}, d_{n,1}, a_{n,0}\}$$

(omitting the $a_{1,1}, d_{0,0}$ for $i = 0$, and omitting the $d_{n,1}, a_{n,0}$ for $i = n$). Let A_i be the corresponding simplex (facet) in the perfect matching complex $\mathcal{M}_p(H_n)$, $i \in \{0, 1, \dots, n\}$. Let $\mathcal{N}(\mathcal{M}_p(H_n))$ be the nerve complex of $\mathcal{M}_p(H_n)$ with vertices A_i . For all i , $0 \leq i \leq n-1$, A_i contains the vertices $d_{n,1}, a_{n,0}$, so $\{A_0, A_1, A_2, \dots, A_{n-1}\}$ is a simplex in $\mathcal{N}(\mathcal{M}_p(H_n))$. For all i , $1 \leq i \leq n$, A_i contains the vertices $a_{1,1}, d_{0,0}$, so $\{A_1, A_2, \dots, A_n\}$ is a simplex in $\mathcal{N}(\mathcal{M}_p(H_n))$. Since $A_0 \cap A_n = \emptyset$ in $\mathcal{M}_p(H_n)$, $\mathcal{N}(\mathcal{M}_p(H_n))$ is the suspension over the simplex $\{A_1, A_2, \dots, A_{n-1}\}$, and so is contractible. The conditions of the nerve lemma are satisfied because a nonempty intersection of simplices in a complex is contractible. Therefore, the perfect matching complex $\mathcal{M}_p(H_n)$ is contractible. \square

3 The $1 \times m \times n$ honeycomb graph

3.1 A short summary of discrete Morse theory

Our subsequent calculations of homotopy type rely on discrete Morse theory. Developed by Forman, discrete Morse theory is a way to find the homotopy type of complexes by pairing faces of the complex [11]. These pairings correspond with a sequence of collapses on the complex, resulting in a homotopy equivalent cell complex.

In what follows we will say two faces are *paired* in place of the usual phrasing of two faces are *matched* in a discrete Morse matching, in order to avoid unnecessary confusion with perfect matchings of a graph.

Definition 5. A **partial pairing** in a poset P is a partial pairing on the underlying graph of the Hasse diagram of P . In other words, it is a subset $M \subseteq P \times P$ such that:

- $(a, b) \in M$ implies $a \prec b$ and
- each $c \in P$ belongs to at most one pair in M .

When $(a, b) \in M$ we write $b = u(a)$. A partial pairing is **acyclic** if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \prec u(a_m) \succ a_1$$

with $m \geq 2$ and $a_i \in P$ distinct.

Given an acyclic partial pairing M on a poset P , we call an element **critical** if it is unpaired. The main theorem of discrete Morse theory describes the essence of these sequences of collapses.

Theorem 6 ([11]). *Let Δ be a polyhedral cell complex and let M be an acyclic pairing on the face poset of Δ . Let c_i denote the number of critical i -dimensional cells of Δ . The space Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single 0-dimensional cell in the case where the empty set is paired in the matching.*

A very simple way of constructing a pairing on a face poset is to choose a vertex and then pair each face that contains that vertex with the subface obtained by deleting that vertex.

Definition 7. [8, 14] Let x be an arbitrary vertex of a simplicial complex K . The **element pairing on K using vertex x** is defined as:

$$M(x) = \{(\sigma, \sigma \cup \{x\}) \mid x \notin \sigma, \sigma \cup \{x\} \in K\}.$$

Throughout the paper we will use the property that a union of a sequence of element pairings is an acyclic pairing, as the following theorem claims.

Theorem 8. ([8, Proposition 2.10], [14, Lemma 4.1]) Let K be a simplicial complex and $\{x_1, x_2, \dots, x_k\}$ be a subset of the vertex set of K . Let $K_0 = K$, and for all $i \in \{1, 2, \dots, k\}$ define inductively:

$$\begin{aligned} M(x_i) &= \{(\sigma, \sigma \cup \{x_i\}) \mid x_i \notin \sigma, \text{ and } \sigma, \sigma \cup \{x_i\} \in K_{i-1}\}, \\ N(x_i) &= \{\sigma \in K_{i-1} \mid \sigma \in \eta \text{ for some } \eta \in M(x_i)\}, \text{ and} \\ K_i &= K_{i-1} \setminus N(x_i). \end{aligned}$$

Then $\bigsqcup_{i=1}^k M(x_i)$ is an acyclic pairing on K .

3.2 Perfect matchings in $H_{1 \times m \times n}$

We label the hexagons and the edges of the $1 \times m \times n$ honeycomb graph as follows. Let $T_{i,j}$ denote the hexagon located in column i ($1 \leq i \leq m$) and row j ($1 \leq j \leq n$) starting from the bottom left hexagon; see Figures 2 and 3. That is, for fixed i , the hexagons $T_{i,j}$, $1 \leq j \leq n$, form a vertical sequence with $T_{i,1}$ on the bottom and $T_{i,n}$ on the top. For fixed j , the hexagons $T_{i,j}$, $1 \leq i \leq m$, form a horizontal sequence with $T_{1,j}$ on the left and $T_{m,j}$ on the right (see Figure 2). The edges are labeled $a_{i,j}$, $b_{i,j}$, $d_{i,j}$, as in Figure 3. Here $a_{i,j}$ exists for $1 \leq i \leq m$ and $0 \leq j \leq n$; $b_{i,j}$ exists for $0 \leq i \leq m$ and $1 \leq j \leq n$; and $d_{i,j}$ exists for $0 \leq i \leq m$ and $0 \leq j \leq n$, except $d_{0,n}$ and $d_{m,0}$.

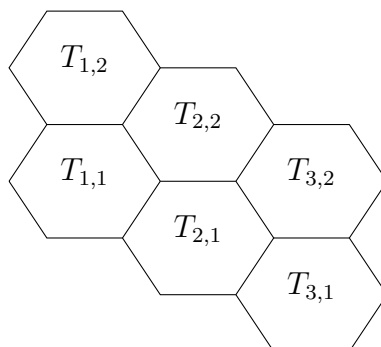


Figure 2: Honeycomb Labels

Remark 9. Recall each perfect matching P on $H_{1 \times m \times n}$ is identified with a plane partition, represented by a $1 \times m$ matrix. Thus, we will denote P as $P = (h_1, h_2, \dots, h_m)$ where $n \geq h_1 \geq \dots \geq h_m \geq 0$ and $h_1, \dots, h_m \in \mathbb{Z}_{\geq 0}$. Notice that we can think of h_1, \dots, h_m as heights of the respective columns of cubes. See Figure 4 for an example.

The matching (h_1, h_2, \dots, h_m) , contains the following edges:

- $a_{i,j}$, if the edge is visible in the top of a cube or if column i has no cubes and $a_{i,j}$ is the bottom horizontal edge in the column;

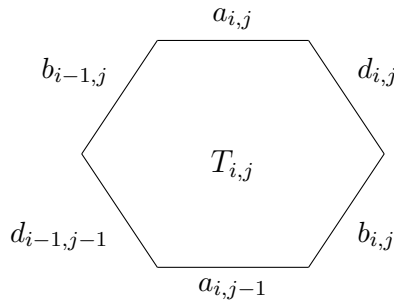


Figure 3: Edge Labels

- $b_{i,j}$, if the edge is visible on the front right side of a cube or $b_{0,j}$ if the hexagon in column 1 is above the top cube in column 1;
- $d_{i,j}$, if the edge is in the front left side of a cube for any column.

In terms of the plane partition, we get the following description of edges in the perfect matching.

Proposition 10. *Let P be the perfect matching corresponding to the plane partition (h_1, h_2, \dots, h_m) . Then*

- $a_{i,j} \in P$ if and only if $j = h_i$;
- $b_{i,j} \in P$ if and only if $(h_i \geq j > h_{i+1})$ or $(i = 0 \text{ and } j > h_1)$ or $(i = m \text{ and } j \leq h_m)$;
- $d_{i,j} \in P$ if and only if $(j > h_i)$ or $(j < h_{i+1})$.

Definition 11. For a $1 \times m \times n$ honeycomb graph, the edges $d_{i,j}$ at the intersection of $T_{i,j}$ and $T_{i+1,j+1}$ for $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$ are called **significant edges**. See Figure 5.

Directly from Definition 1 we derive the following properties of facets of $\mathcal{M}_p(H_{1 \times m \times n})$.

Lemma 12. *Consider the honeycomb graph $H = H_{1 \times m \times n}$, $m, n \in \mathbb{N}$, $m, n \geq 2$, and its perfect matching complex $\mathcal{M}_p(H)$. Consider vertices of $\mathcal{M}_p(H)$ that correspond to edges $x = d_{0,0}$ and $y = d_{m,n}$ (Figure 5). Let $P = (h_1, \dots, h_m)$ be an arbitrary perfect matching on H . Then vertex x belongs to P if and only if $P \neq \underbrace{(0, \dots, 0)}_m$, while vertex y belongs to P if and only if $P \neq \underbrace{(n, \dots, n)}_m$.*

Proof. By Proposition 10, the vertex $y = d_{m,n}$ belongs to P if and only if $h_m < n$, which occurs if and only if $P \neq \underbrace{(n, \dots, n)}_m$. The vertex $x = d_{0,0}$ belongs to P if and only if $h_1 > 0$, which occurs if and only if $P \neq \underbrace{(0, \dots, 0)}_m$. □

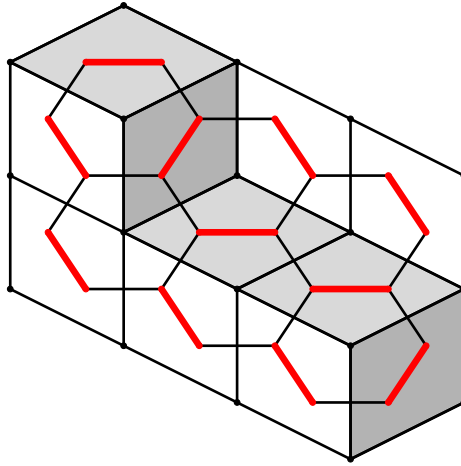


Figure 4: Pictured above is $H_{1 \times 3 \times 2}$ along with a perfect matching highlighted in thicker red edges and overlaid with the corresponding plane partition $(2, 1, 1)$.

Lemma 13. Consider the honeycomb graph $H = H_{1 \times m \times n}$, $m, n \in \mathbb{N}$, $m, n \geq 2$, and its perfect matching complex $\mathcal{M}_p(H)$. Then the intersection of facet $\underbrace{(0, \dots, 0)}_m$ and facet $\underbrace{(n, \dots, n)}_m$ is a simplex whose set of vertices corresponds to the set of significant edges.

Proof. If P equals either $(0, \dots, 0)$ or (n, \dots, n) , then, according to Proposition 10, for $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$, P contains neither $a_{i,j}$ nor $b_{i,j+1}$. Each interior vertex of the honeycomb graph lies on exactly three edges, $a_{i,j}$, $b_{i,j+1}$ and $d_{i,j}$ ($1 \leq i \leq m - 1$, $1 \leq j \leq n - 1$). Thus, to cover this interior vertex, P must contain the significant edge $d_{i,j}$.

We check that the two perfect matchings share no other edges. Clearly, they share no edge $a_{i,j}$. The matching $(0, \dots, 0)$ contains $b_{0,j}$ ($1 \leq j \leq n$) but no other $b_{i,j}$. The matching (n, \dots, n) contains $b_{m,j}$ ($1 \leq j \leq n$) but no other $b_{i,j}$. So the two share no edge $b_{i,j}$. Finally, (n, \dots, n) contains no $d_{i,n}$ and no $d_{m,j}$, while $(0, \dots, 0)$ contains no $d_{0,j}$ and no $d_{i,0}$. So the intersection of the perfect matchings $(0, \dots, 0)$ and (n, \dots, n) is exactly the simplex with vertices corresponding to the significant edges. \square

We will use this Lemma in the next two sections as we create Morse pairings.

3.3 The $1 \times 2 \times n$ honeycomb graph

Before proving the main result for $1 \times m \times n$ honeycomb graphs, we consider the homotopy type in the special case $H_{1 \times 2 \times n}$.

Theorem 14. Let $H_{1 \times 2 \times n}$ be the honeycomb graph with $n \geq 2$. Then

$$\mathcal{M}_p(H_{1 \times 2 \times n}) \simeq S^{n-1}.$$

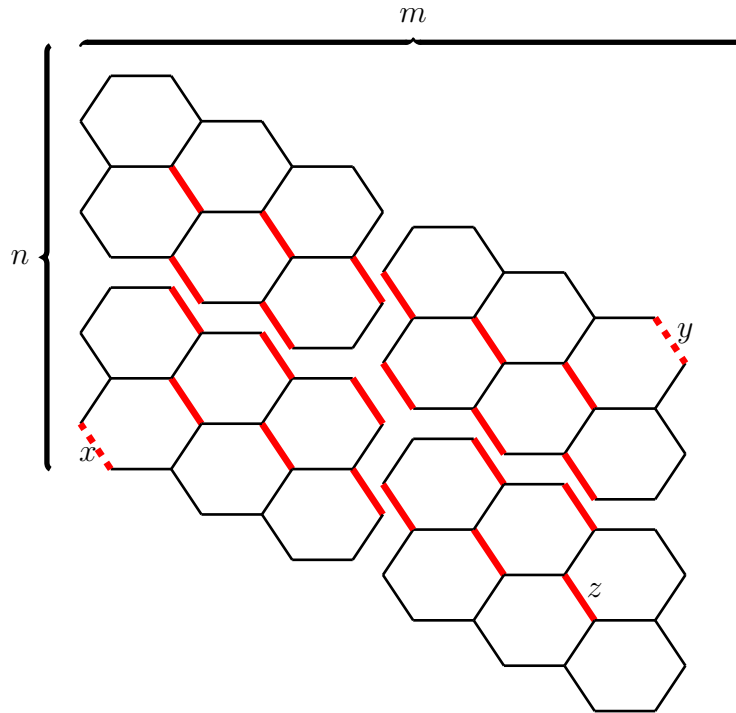


Figure 5: The $1 \times m \times n$ honeycomb graph with significant edges highlighted in thicker red.

Proof. Let $H = H_{1 \times 2 \times n}$. We construct a Morse pairing on the face poset of $\mathcal{M}_p(H)$ by using two element pairings:

- (1) construct element pairing $M(x)$ using vertex x (see Figure 6); then
- (2) on the set of unpaired faces, construct element pairing $M(y)$ using vertex y .

From Theorem 8, we know that the union of a sequence of element pairings is an acyclic pairing. Therefore, $M(x) \cup M(y)$ is an acyclic pairing on the face poset of $\mathcal{M}_p(H)$. We will prove that there is only one face of $\mathcal{M}_p(H)$ that is unpaired after $M(x) \cup M(y)$.

From Lemma 12, we know that vertex x belongs to a perfect matching $P = (h_1, h_2)$ if and only if $P \neq (0, 0)$, while vertex y belongs to P if and only if $P \neq (n, n)$. Therefore, complex $\mathcal{M}_p(H)$ can be seen as the union of two cones with apices x and y . From this representation we observe that a face $\tau \in \mathcal{M}_p(H)$ is unpaired after $M(x) \cup M(y)$ if and only if τ has the following structure:

$$\tau = \{y\} \cup \sigma,$$

where the face $\sigma \in \mathcal{M}_p(H)$ satisfies:

$$x, y \notin \sigma, \{y\} \cup \sigma \in \mathcal{M}_p(H), \{x\} \cup \sigma \in \mathcal{M}_p(H), \text{ and } \{x, y\} \cup \sigma \notin \mathcal{M}_p(H). \quad (1)$$

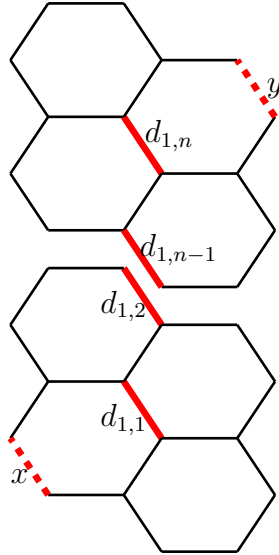


Figure 6: Case $1 \times 2 \times n$.

Let $\sigma \in \mathcal{M}_p(H)$ be an arbitrary face that satisfies (1). If there exists a perfect matching $P \notin \{(0,0), (n,n)\}$ such that $\sigma \subseteq P$, then $\sigma \cup \{x,y\}$ is also contained in P , which contradicts the last condition in (1). The only perfect matchings that might contain σ are $(0,0)$ and (n,n) . Further, since $x \notin (0,0)$ and $\{x\} \cup \sigma \in \mathcal{M}_p(H)$, we conclude that $\sigma \subseteq (n,n)$. Similarly, conditions $y \notin (n,n)$ and $\{y\} \cup \sigma \in \mathcal{M}_p(H)$ imply that $\sigma \subseteq (0,0)$. Therefore,

$$\sigma \subseteq (0,0) \cap (n,n) = \{d_{1,1}, \dots, d_{1,n-1}\},$$

where the second equality follows from Lemma 13.

Suppose that $\rho \subsetneq \{d_{1,1}, \dots, d_{1,n-1}\}$ is a proper subset. Then, there exists some index $i \in \{1, \dots, n-1\}$ such that $d_{1,i} \notin \rho$. The perfect matching $(i,i) \notin \{(0,0), (n,n)\}$ contains $\{d_{1,1}, d_{1,2}, \dots, d_{1,n-1}\} \setminus \{d_{1,i}\}$. Therefore (i,i) is a perfect matching that contains ρ .

Hence, there is only one critical cell, $\tau = \{y\} \cup \{d_{1,1}, \dots, d_{1,n-1}\}$. From Theorem 6, complex $\mathcal{M}_p(H)$ is homotopy equivalent to a CW-complex with one $(n-1)$ -dimensional cell and one 0-cell (because the empty set is paired with set $\{x\}$). Hence, $\mathcal{M}_p(H) \simeq \mathcal{S}^{n-1}$. \square

3.4 The $1 \times m \times n$ honeycomb graph, $m, n \geq 3$

The homotopy type for $H_{1 \times m \times n}$ is different when m and n are both at least 3. In the proof of the following theorem, we emphasize two places where we use the fact that $m, n \geq 3$.

Theorem 15. *Let $H_{1 \times m \times n}$ be the honeycomb graph of dimension $1 \times m \times n$, with $m, n \in \mathbb{N}$, and $m, n \geq 3$. Then the perfect matching complex $\mathcal{M}_p(H_{1 \times m \times n})$ is contractible.*

Proof. Let $H = H_{1 \times m \times n}$. We construct a discrete Morse pairing for $\mathcal{M}_p(H)$. Begin with the two element pairings as in Theorem 14, and extend it with one significant edge:

- (1) construct an element pairing $M(x)$ using vertex $x = d_{0,0}$; then
- (2) on the set of unpaired faces, construct element pairing $M(y)$ using vertex $y = d_{m,n}$; then
- (3) on the set of unpaired faces, construct element pairing $M(d_{m-1,1})$ using vertex $z = d_{m-1,1}$.

(See Figure 5 for edge labels.)

From Theorem 8, we know that the union of a sequence of element pairings is an acyclic pairing; therefore $M(x) \cup M(y) \cup M(z)$ is an acyclic pairing on the face poset of $\mathcal{M}_p(H)$.

As in the proof of Theorem 14, we see that a face $\tau \in \mathcal{M}_p(H)$ is unpaired after $M(x) \cup M(y)$ if and only if τ has the following structure:

$$\tau = \{y\} \cup \sigma,$$

where $\sigma \subseteq (0, \dots, 0) \cap (n, \dots, n)$ and $\tau \cup \{x\} \notin \mathcal{M}_p(H)$.

Let N be the set of faces in $\mathcal{M}_p(H)$ that are unpaired after $M(x) \cup M(y)$. Reasoning as in the proof of Theorem 14 we get that

$$N = \{\tau \in \mathcal{M}_p(H) : \tau = \{y\} \cup \sigma \text{ for some } \sigma \subseteq (0, \dots, 0) \cap (n, \dots, n) \text{ and } \tau \cup \{x\} \notin \mathcal{M}_p(H)\}.$$

Note that $\sigma \subseteq (0, \dots, 0) \cap (n, \dots, n)$ implies that $x, y \notin \sigma$. We claim that the final element pairing $M(z)$ pairs all faces in N , and therefore the homotopy type of the perfect matching complex is contractible. Thus, we wish to show $z \notin \tau$, $\tau \in N$ if and only if $\tau \cup \{z\} \in N$.

(\Rightarrow) Assume $\tau = \{y\} \cup \sigma \in N$, $z \notin \tau$. Then $\sigma \subseteq (0, \dots, 0) \cap (n, \dots, n)$ and $\tau \cup \{x\} \notin \mathcal{M}_p(H)$. Consider $\tau \cup \{z\} = \{y\} \cup (\sigma \cup \{z\})$. Since $z \in (0, \dots, 0) \cap (n, \dots, n)$, as z is a significant edge, $\sigma \cup \{z\} \subseteq (0, \dots, 0) \cap (n, \dots, n)$. This also shows that $\tau \cup \{z\} \subseteq (0, \dots, 0)$, and hence is in $\mathcal{M}_p(H)$. Also, $(\tau \cup \{z\}) \cup \{x\} \notin \mathcal{M}_p(H)$, since its subset $\tau \cup \{x\}$ is not in $\mathcal{M}_p(H)$. So $\tau \cup \{z\} \in N$.

(\Leftarrow) Assume $\tau \cup \{z\} \in N$ ($z \notin \tau$). Then $\tau \cup \{z\} = \{y\} \cup (\sigma \cup \{z\})$ where $\sigma \cup \{z\} \subseteq (0, \dots, 0) \cap (n, \dots, n)$ and $(\tau \cup \{z\}) \cup \{x\} \notin \mathcal{M}_p(H)$. Clearly, $\tau = \{y\} \cup \sigma \in \mathcal{M}_p(H)$, with $\sigma \subseteq (0, \dots, 0) \cap (n, \dots, n)$. We need to show that $\tau \cup \{x\} = \sigma \cup \{x, y\} \notin \mathcal{M}_p(H)$. We prove this by contradiction.

Assume $\sigma \cup \{x, y\} \in \mathcal{M}_p(H)$, but $(\sigma \cup \{z\}) \cup \{x, y\} \notin \mathcal{M}_p(H)$. Say $\sigma \cup \{x, y\} \subseteq (h_1, h_2, \dots, h_m)$. Since $(\sigma \cup \{z\}) \cup \{x, y\} \not\subseteq (h_1, h_2, \dots, h_m)$, by Proposition 10 (applied to $z = d_{m-1,1}$), $h_{m-1} \geq 1$ and $h_m \leq 1$.

Case 1. $h_{m-1} \geq 2$ and $h_m \leq 1$. We claim that in this case $(\sigma \cup \{z\}) \cup \{x, y\} \subseteq (h_1, h_2, \dots, h_{m-1}, 2)$. First note that x and y are in $(h_1, h_2, \dots, h_{m-1}, 2)$, since each is in every perfect matching except $(0, \dots, 0)$ (in the case of x) and (n, \dots, n) (in the case of y). Here we use the fact $n \geq 3$, which implies $y \in (h_1, h_2, \dots, h_{m-1}, 2)$. Since $\sigma \subseteq (0, \dots, 0) \cap (n, \dots, n)$, all other elements of $(\sigma \cup \{z\}) \cup \{x, y\}$ are significant edges, that is, edges of

the form $d_{i,j}$, $1 \leq i \leq m-1$, $1 \leq j \leq n-1$. By Proposition 10, $d_{i,j} \in (h_1, h_2, \dots, h_{m-1}, 2)$ if and only if $j > h_i$ or $j < h_{i+1}$. (We are assuming $i > 0$.) In particular, $1 < 2$, so $z = d_{m-1,1} \in (h_1, \dots, h_{m-1}, 2)$. In addition, we know $\sigma \cup \{x, y\} \subseteq (h_1, h_2, \dots, h_m)$, so for $d_{i,j} \in \sigma$ with $i < m-1$, $d_{i,j}$ satisfies the criterion for $(h_1, h_2, \dots, h_{m-1}, 2)$. On the other hand, for $d_{m-1,j} \in \sigma$, $j \geq 2 > h_m$, so $d_{m-1,j} \in (h_1, h_2, \dots, h_m)$ implies $j > h_{m-1}$. Thus, $d_{m-1,j}$ also satisfies the criterion for $(h_1, h_2, \dots, h_{m-1}, 2)$. So in Case 1, we conclude that $(\tau \cup \{z\}) \cup \{x\} = (\sigma \cup \{z\}) \cup \{x, y\} \in \mathcal{M}_p(H)$, a contradiction.

Case 2. $h_{m-1} = 1$ and $h_m \leq 1$. We claim that in this case $(\sigma \cup \{z\}) \cup \{x, y\} \subseteq (h_1, h_2, \dots, h_{m-2}, 0, 0)$. As in Case 1, x and y are in $(h_1, h_2, \dots, h_{m-2}, 0, 0)$. Note that since $m \geq 3$ and $h_{m-1} = 1$, we have $h_{m-2} \geq 1$. Also, $z = d_{m-1,1}$ is in $(h_1, h_2, \dots, h_{m-2}, 0, 0)$, since $1 > 0$. Again, consider the significant edges $d_{i,j}$ in σ . We know these $d_{i,j}$ are in $(h_1, h_2, \dots, h_{m-1}, 1, h_m)$, so for $i < m-2$ it is obvious that $d_{i,j} \in (h_1, h_2, \dots, h_{m-2}, 0, 0)$. For $i = m-2$, the condition from Proposition 10 reduces to $j > h_{m-2}$, so $d_{m-2,j} \in (h_1, h_2, \dots, h_{m-2}, 0, 0)$ again. Finally, for $i = m-1$, Proposition 10 implies $j > 1 > 0$, and $d_{i,j} \in (h_1, h_2, \dots, h_{m-2}, 0, 0)$.

Again we have shown that $(\tau \cup \{z\}) \cup \{x\} = (\sigma \cup \{z\}) \cup \{x, y\} \in \mathcal{M}_p(H)$, a contradiction.

We conclude that $\tau \cup \{x\} = \sigma \cup \{x, y\} \notin \mathcal{M}_p(H)$. So $\tau \in N$.

Thus, all elements unpaired after $M(x) \cup M(y)$ are paired as $(\tau, \tau \cup \{z\})$. That is, the three element pairings, $M(x)$, $M(y)$, and $M(z)$ pair all faces of $\mathcal{M}_p(H_{1 \times m \times n})$, so $\mathcal{M}_p(H_{1 \times m \times n})$ ($m, n \geq 3$) is contractible. \square

4 The $2 \times 2 \times 2$ honeycomb graph

We conclude this article by calculating the homotopy type of the perfect matching complex for the $2 \times 2 \times 2$ honeycomb graph. Recall that each perfect matching on the honeycomb graph is in bijection with a plane partition. For the $2 \times 2 \times 2$ honeycomb graph that means we are considering plane partitions of shape $(2, 2)$. We will denote these plane partitions by $\frac{a}{c} \Big| \frac{b}{d}$ where $0 \leq a, b, c, d \leq 2$ and $a \geq b, c \geq d$. For an example of the

plane partition represented by $\frac{2}{1} \Big| \frac{2}{1}$, see Figure 7. See Appendix A for drawings of all the perfect matchings of $H_{2 \times 2 \times 2}$ and the corresponding plane partitions.

Let $\sigma \in \mathcal{M}_p(H)$ be a face in the perfect matching complex of the $2 \times 2 \times 2$ honeycomb graph H . We use the notation $\sigma \in \frac{a}{c} \Big| \frac{b}{d}$ to denote a subset of the perfect matching

corresponding with $\frac{a}{c} \Big| \frac{b}{d}$. If an entry in the plane partition can be 0, 1, or 2, in accordance with the restrictions, we will denote it with $*$. For example, $\frac{2}{*} \Big| \frac{2}{1}$ represents the

plane partitions $\frac{2}{1} \Big| \frac{2}{1}$ and $\frac{2}{2} \Big| \frac{2}{1}$.

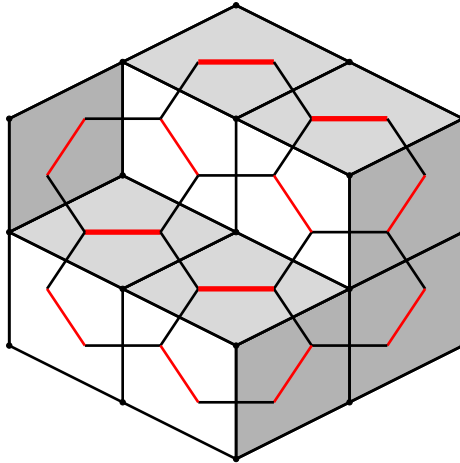


Figure 7: Matching for plane partition

Remark 16. In the proof that follows we will use a sequence of element pairings to obtain the homotopy type. Notice that if we perform a sequence of element pairings in which an element ε has been paired on, we can categorize the remaining faces as those that contain ε and those that do not. When we continue pairing with an element that has not previously been paired on, say λ , we can only pair faces of the same type. That is to say θ and $\theta \cup \lambda$ would be paired together only if ε was in θ and $\theta \cup \lambda$, or ε was not in θ and $\theta \cup \lambda$.

Theorem 17. *Let H be the honeycomb graph of dimension $2 \times 2 \times 2$. Then,*

$$\mathcal{M}_p(H) \simeq S^3 \vee S^3.$$

Proof. We will proceed by defining a discrete Morse matching on $\mathcal{M}_p(H)$ given by pairing on α , then β , then γ , and finally δ according to the labels in Figure 8.

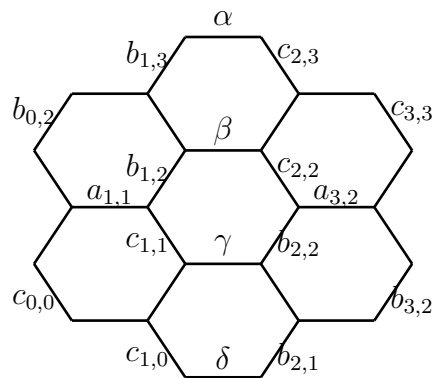


Figure 8: Figure for the proof of Theorem 17.

For faces $\sigma \in \mathcal{M}_p(H)$, we make the following observations:

- (1) $\sigma \cup \{\alpha\}$ is a face if and only if $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$,
- (2) $\sigma \cup \{\beta\}$ is a face if and only if $\sigma \in \frac{2}{2} \Big| \frac{2}{2}$ or $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$,
- (3) $\sigma \cup \{\gamma\}$ is a face if and only if $\sigma \in \frac{0}{0} \Big| \frac{0}{0}$ or $\sigma \in \frac{*}{*} \Big| \frac{*}{1}$, and
- (4) $\sigma \cup \{\delta\}$ is a face if and only if $\sigma \in \frac{*}{*} \Big| \frac{*}{0}$.

We begin by an element pairing on α and then β . There are two types of unpaired simplices that remain. The first type (t1) consists of faces $\sigma \in \mathcal{M}_p(H)$ such that $\alpha, \beta \notin \sigma$, and σ cannot be paired with $\sigma \cup \alpha$ or $\sigma \cup \beta$, because simplices $\sigma \cup \alpha$ and $\sigma \cup \beta$ do not exist in $\mathcal{M}_p(H)$. Therefore, by observations (1) and (2) $\sigma \notin \frac{2}{*} \Big| \frac{*}{*}$, $\sigma \notin \frac{2}{2} \Big| \frac{2}{2}$, and $\sigma \notin \frac{1}{*} \Big| \frac{*}{*}$. It follows that σ is only in $\frac{0}{0} \Big| \frac{0}{0}$ and $\alpha, \beta \notin \sigma$.

The second type (t2) consists of faces $\sigma \cup \beta \in \mathcal{M}_p(H)$, which remain because σ has been previously paired with $\sigma \cup \alpha$, and $\sigma \cup \beta \cup \alpha \notin \mathcal{M}_p(H)$. There exists a face $\sigma \cup \alpha$, so $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$. Similarly, $\sigma \cup \beta$ is a face, so $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$ or $\sigma \in \frac{2}{2} \Big| \frac{2}{2}$. But $\sigma \cup \beta \cup \alpha$ is not a face, so σ is not in $\frac{2}{2} \Big| \frac{2}{2}$. Therefore, (t2) is the set of faces $\sigma \cup \beta$ such that $\alpha, \beta \notin \sigma$, $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$ and $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$, and $\sigma \notin \frac{2}{2} \Big| \frac{2}{2}$.

By Remark 16, we know that in the remaining element pairings if two faces are paired they have to be of the same type. We perform our next pairing with γ and analyze what faces remain unpaired. For each of the above types we will consider faces that contain γ and those that do not.

Let $\theta \in \mathcal{M}_p(H)$.

Case 1: Suppose θ is of type (t1) and $\gamma \notin \theta$. Since θ is type (t1), θ is only in $\frac{0}{0} \Big| \frac{0}{0}$ but, it is also the case that $\gamma \in \frac{0}{0} \Big| \frac{0}{0}$. This means that $\theta \cup \gamma$ is only in $\frac{0}{0} \Big| \frac{0}{0}$, so θ and $\theta \cup \gamma$ are paired and there are no unpaired faces that remain from this case.

Case 2: Suppose $\theta \cup \gamma$ is of type (t1). Then $\theta \cup \gamma$ is only in $\frac{0}{0} \Big| \frac{0}{0}$. The only way this face could be unpaired is if θ is not only in $\frac{0}{0} \Big| \frac{0}{0}$. Hence the faces that remain are all $\theta \cup \gamma$ such that $\alpha, \beta, \gamma \notin \theta$ and $\theta \cup \gamma$ is only in $\frac{0}{0} \Big| \frac{0}{0}$ and either $\theta \in \frac{2}{*} \Big| \frac{*}{*}$ or $\theta \in \frac{1}{*} \Big| \frac{*}{*}$. We call these faces types (t1.1).

Case 3: Suppose θ is of type (t2) and $\gamma \notin \theta$. Since θ is of type (t2), $\theta = \sigma \cup \beta$ where $\alpha, \beta \notin \sigma$ and $\sigma \in \frac{2}{*} \left| \frac{*}{*} \right.$, and $\sigma \in \frac{1}{*} \left| \frac{*}{*} \right.$, and $\sigma \notin \frac{2}{2} \left| \frac{2}{2} \right.$. Since we are supposing that θ is unpaired, $\theta \cup \gamma = \sigma \cup \gamma \cup \beta$ is not of type (t2). Therefore, $\sigma \cup \gamma \notin \frac{2}{*} \left| \frac{*}{*} \right.$ or $\sigma \cup \gamma \notin \frac{1}{*} \left| \frac{*}{*} \right.$ or $\sigma \cup \gamma \in \frac{2}{2} \left| \frac{2}{2} \right.$. Notice the last condition is not possible since $\gamma \notin \frac{2}{2} \left| \frac{2}{2} \right.$ and, since γ is in a face if and only if the face is in $\frac{*}{*} \left| \frac{*}{1} \right.$ or in $\frac{0}{0} \left| \frac{0}{0} \right.$, we can rewrite the above statement. That is, since $\theta \cup \gamma$ is not of type (t2), $\sigma \notin \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \notin \frac{1}{1} \left| \frac{1}{1} \right.$. Hence the unpaired faces that remain are $\sigma \cup \beta$ where $\alpha, \beta, \gamma \notin \sigma$, $\sigma \in \frac{2}{*} \left| \frac{*}{*} \right.$ and $\sigma \in \frac{1}{*} \left| \frac{*}{*} \right.$ and $\sigma \notin \frac{2}{2} \left| \frac{2}{2} \right.$ and either σ is not in $\frac{2}{*} \left| \frac{*}{1} \right.$ or σ is not in $\frac{1}{1} \left| \frac{1}{1} \right.$. We call this type (t2.1).

Case 4: Suppose $\theta \cup \gamma$ is of type (t2). Then, $\theta \cup \gamma = \sigma \cup \beta \cup \gamma$ and, as argued in Case 3, $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$. Since we are assuming that $\theta \cup \gamma = \sigma \cup \gamma \cup \beta$ is unpaired this must be because $\theta = \sigma \cup \beta$ is not of type (t2). This implies that $\theta \notin \frac{2}{*} \left| \frac{*}{*} \right.$, which cannot be the case because $\sigma \cup \gamma \in \frac{2}{*} \left| \frac{*}{1} \right.$, or $\sigma \notin \frac{1}{*} \left| \frac{*}{*} \right.$, which cannot be the case because $\sigma \cup \gamma \in \frac{1}{1} \left| \frac{1}{1} \right.$, or $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$. It follows that, for this case, all unpaired faces are $\sigma \cup \beta \cup \gamma$ such that $\alpha, \beta, \gamma \notin \sigma$ and $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$ and $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$. We call this type (t2.2).

We are now ready to pair using δ . Consider first the faces of type (t2.2).

Case 4.1: Suppose θ is of type (t2.2) and $\delta \in \theta$. Then $\delta \in \sigma$. This case cannot occur because in type (t2.2) $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$ and $\delta \notin \frac{1}{1} \left| \frac{1}{1} \right.$.

Case 4.2: Suppose θ is of type (t2.2) and $\delta \notin \theta$. That is, $\theta = \sigma \cup \beta \cup \gamma$ such that $\alpha, \beta, \gamma, \delta \notin \sigma$ and $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$ and $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$. For the faces left unpaired, it must be the case that $\sigma \cup \delta \cup \beta \cup \gamma$ is not of type (t2.2). That is, $\sigma \cup \delta \notin \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \cup \delta \notin \frac{1}{1} \left| \frac{1}{1} \right.$ or $\sigma \cup \delta \notin \frac{2}{2} \left| \frac{2}{2} \right.$, which is true since $\delta \notin \frac{1}{1} \left| \frac{1}{1} \right.$. Therefore, the unpaired cells are $\sigma \cup \beta \cup \gamma$ such that $\alpha, \beta, \gamma, \delta \notin \sigma$ and $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$ and $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$.

When we consider the intersection $\frac{2}{*} \left| \frac{*}{1} \right. \cap \frac{1}{1} \left| \frac{1}{1} \right. \cap \frac{2}{2} \left| \frac{2}{2} \right.$ we have $c_{0,0}, c_{1,0}, b_{2,1}$, and $b_{3,2}$. See Figure 9. These four edges are in the intersection $\frac{2}{2} \left| \frac{2}{1} \right. \cap \frac{1}{1} \left| \frac{1}{1} \right. \cap \frac{2}{2} \left| \frac{2}{2} \right.$. One can verify that these are the only edges in the intersection $\frac{2}{*} \left| \frac{*}{1} \right. \cap \frac{1}{1} \left| \frac{1}{1} \right. \cap \frac{2}{2} \left| \frac{2}{2} \right.$.

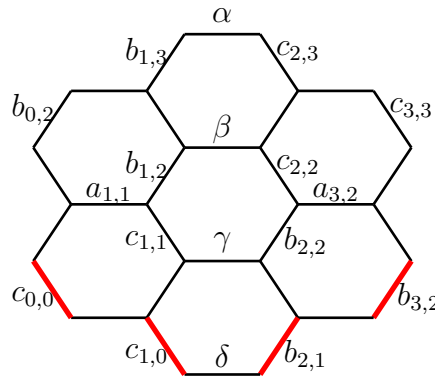


Figure 9: The highlighted edges used in Case 4.2.

Therefore, $\sigma \in \mathcal{P}(c_{0,0}, c_{1,0}, b_{2,1}, b_{3,2})$, the power set of $\{c_{0,0}, c_{1,0}, b_{2,1}, b_{3,2}\}$. So the faces θ in Case 4.2 form an interval in the face poset of $\mathcal{M}_p(H)$ isomorphic to the Boolean lattice, and so can be paired using an element pairing with, say, $c_{1,0}$.

We proceed with the analysis of element pairing using δ for unpaired faces of type (t1.1) from Case 2.

Case 2.1: Suppose θ is of type (t1.1) and $\delta \notin \theta$. That is, $\theta = \sigma \cup \gamma$ such that $\alpha, \beta, \gamma, \delta \notin \sigma$, $\sigma \cup \gamma$ is only in $\frac{0}{0} \left| \frac{0}{0} \right.$ and $\sigma \in \frac{2}{*} \left| \frac{*}{*} \right.$ or $\sigma \in \frac{1}{*} \left| \frac{*}{*} \right.$. For the faces left unpaired, it must be that $\sigma \cup \delta \cup \gamma$ is not of type (t1.1). Therefore, it is either the case that $\sigma \cup \delta \cup \gamma$ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$ or that $\sigma \cup \delta \notin \frac{2}{*} \left| \frac{*}{*} \right.$ and $\sigma \cup \delta \notin \frac{1}{*} \left| \frac{*}{*} \right.$, but notice if $\sigma \cup \delta \cup \gamma$ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$ then $\sigma \cup \gamma$ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$, which is a contradiction to θ being type (t1.1). Thus, it must be the case that $\sigma \cup \delta \notin \frac{2}{*} \left| \frac{*}{*} \right.$ and $\sigma \cup \delta \notin \frac{1}{*} \left| \frac{*}{*} \right.$ and, in particular, $\sigma \notin \frac{2}{*} \left| \frac{*}{0} \right.$ and $\sigma \notin \frac{1}{*} \left| \frac{*}{0} \right.$. It follows that the faces left unpaired are those such that $\sigma \cup \gamma$ is only in $\frac{0}{0} \left| \frac{0}{0} \right.$ and ($\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$ or $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$). Notice it cannot be the case that $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$ because $\sigma \cup \gamma$ is only in $\frac{0}{0} \left| \frac{0}{0} \right.$. So we are left with $\sigma \cup \gamma$ only in $\frac{0}{0} \left| \frac{0}{0} \right.$ and $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$, but this implies that

σ is an empty face, because $\frac{0}{0} \left| \frac{0}{0} \right. \cap \frac{2}{2} \left| \frac{2}{2} \right. = \{\emptyset\}$. Therefore, $\sigma \cup \gamma = \gamma$, and this is a contradiction because γ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$. Hence, there are no unpaired faces for this case.

Case 2.2: Suppose θ is of type (t1.1) and $\delta \in \theta$. That is, $\theta = \sigma \cup \delta \cup \gamma$ where $\alpha, \beta, \gamma, \delta \notin \sigma$, $\sigma \cup \delta \cup \gamma$ is only in $\frac{0}{0} \left| \frac{0}{0} \right.$ and $\sigma \cup \delta \in \frac{2}{*} \left| \frac{*}{*} \right.$ or $\sigma \cup \delta \in \frac{1}{*} \left| \frac{*}{*} \right.$. For the faces left unpaired, it must be that $\sigma \cup \gamma$ is not of type (t1.1). Therefore, it is either the case that $\sigma \cup \gamma$ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$ or $\sigma \notin \frac{2}{*} \left| \frac{*}{*} \right.$ and $\sigma \notin \frac{1}{*} \left| \frac{*}{*} \right.$, but $\sigma \cup \delta$ is already in $\frac{2}{*} \left| \frac{*}{*} \right.$ or $\frac{1}{*} \left| \frac{*}{*} \right.$. Therefore, it must be that $\sigma \cup \gamma$ is not only in $\frac{0}{0} \left| \frac{0}{0} \right.$. Hence all unpaired faces of this type are such that $\sigma \in \frac{0}{0} \left| \frac{0}{0} \right.$ and ($\sigma \in \frac{2}{*} \left| \frac{*}{0} \right.$ or $\sigma \in \frac{1}{*} \left| \frac{*}{0} \right.$) and ($\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$).

These matchings σ are subsets of the highlighted edges in Figure 10. It can be checked that no other edges are in a matching σ of this type. The edges $b_{0,2}, b_{1,3}, c_{2,3}$, and $c_{3,3}$ are all in $\frac{0}{0} \left| \frac{0}{0} \right. \cap \frac{1}{1} \left| \frac{1}{0} \right. \cap \frac{1}{1} \left| \frac{1}{1} \right.$. So the unpaired faces $\sigma \cup \delta \cup \gamma$ include those for which

$\sigma \in P(b_{0,2}, b_{1,3}, c_{2,3}, c_{3,3})$. The edges $b_{1,2}$ and $c_{2,2}$ are both in $\frac{0}{0} \left| \frac{0}{0} \right. \cap \frac{2}{2} \left| \frac{2}{0} \right. \cap \frac{2}{2} \left| \frac{2}{1} \right.$.

The edges $b_{1,2}$ and $c_{3,3}$ are both in $\frac{0}{0} \left| \frac{0}{0} \right. \cap \frac{2}{2} \left| \frac{0}{0} \right. \cap \frac{2}{2} \left| \frac{1}{1} \right.$. The edges $b_{0,2}$ and $c_{2,2}$ are

both in $\frac{0}{0} \left| \frac{0}{0} \right. \cap \frac{2}{0} \left| \frac{2}{0} \right. \cap \frac{2}{1} \left| \frac{2}{1} \right.$. So the unpaired faces $\sigma \cup \delta \cup \gamma$ include those for which

$\sigma \in \{\{b_{1,2}\}, \{c_{2,2}\}, \{b_{1,2}, c_{2,2}\}, \{b_{1,2}, c_{3,3}\}, \{b_{0,2}, c_{2,2}\}\}$. It is straightforward to check that these describe all the unpaired faces. Thus, the unpaired faces in this case are $\sigma \cup \delta \cup \gamma$ where $\sigma \in P(b_{0,2}, b_{1,3}, c_{3,3}, c_{2,3}) \cup \{\{b_{1,2}\}, \{c_{2,2}\}, \{b_{1,2}, c_{2,2}\}, \{b_{1,2}, c_{3,3}\}, \{b_{0,2}, c_{2,2}\}\}$.

Finally, we discuss Case 3 with element pairing using δ .

Case 3.1: Suppose θ is of type (t2.1) and $\delta \notin \theta$. That is, $\theta = \sigma \cup \beta$ where $\alpha, \beta, \gamma, \delta \notin \sigma$, $\sigma \in \frac{2}{*} \left| \frac{*}{*} \right.$, $\sigma \in \frac{1}{*} \left| \frac{*}{*} \right.$, $\sigma \notin \frac{2}{2} \left| \frac{2}{2} \right.$, and ($\sigma \notin \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \notin \frac{1}{1} \left| \frac{1}{1} \right.$). If

$\theta \cup \delta = \sigma \cup \delta \cup \beta$ is also of type (t2.1), then θ and $\theta \cup \delta$ are paired using δ . So assume $\sigma \cup \delta \cup \beta$ is not of type (t2.1). Then $\sigma \cup \delta \notin \frac{2}{*} \left| \frac{*}{*} \right.$ or $\sigma \cup \delta \notin \frac{1}{*} \left| \frac{*}{*} \right.$ or $\sigma \cup \delta \in \frac{2}{2} \left| \frac{2}{2} \right.$

or ($\sigma \cup \delta \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \cup \delta \in \frac{1}{1} \left| \frac{1}{1} \right.$). Note that by observation (4) $\sigma \cup \delta$ cannot be

in $\frac{2}{2} \left| \frac{2}{2} \right.$ or $\frac{1}{1} \left| \frac{1}{1} \right.$. Therefore, we see that all unpaired faces are such that $\sigma \in \frac{2}{*} \left| \frac{*}{*} \right.$,

and $\sigma \in \frac{1}{*} \left| \frac{*}{*} \right.$ and $\sigma \notin \frac{2}{2} \left| \frac{2}{2} \right.$, and ($\sigma \notin \frac{2}{*} \left| \frac{*}{1} \right.$ or $\sigma \notin \frac{1}{1} \left| \frac{1}{1} \right.$) and ($\sigma \notin \frac{2}{*} \left| \frac{*}{0} \right.$ or

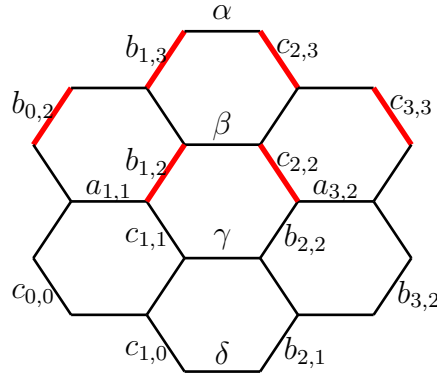


Figure 10: The highlighted edges used in Case 2.2

$\sigma \notin \frac{1}{*} \Big| \frac{*}{0}$). Although there are four possible subcases to consider, we see that it is not possible for $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$, $\sigma \notin \frac{2}{2} \Big| \frac{2}{2}$, $\sigma \notin \frac{2}{*} \Big| \frac{*}{1}$, and $\sigma \notin \frac{2}{*} \Big| \frac{*}{0}$, which rules out one subcase. Similarly, it is not possible for $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$, $\sigma \notin \frac{1}{1} \Big| \frac{1}{1}$, and $\sigma \notin \frac{1}{*} \Big| \frac{*}{0}$. So we are left with two possible subcases, both of which lead to no unpaired faces.

Subcase 1: Suppose that $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$, $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$, $\sigma \notin \frac{2}{2} \Big| \frac{2}{2}$, $\sigma \notin \frac{2}{*} \Big| \frac{*}{1}$, and $\sigma \notin \frac{1}{*} \Big| \frac{*}{0}$. Then, it follows that $\sigma \in \frac{2}{*} \Big| \frac{*}{0}$ and $\sigma \in \frac{1}{1} \Big| \frac{1}{1}$. We now notice that the intersection of the perfect matchings of $\frac{1}{1} \Big| \frac{1}{1}$ and $\frac{2}{*} \Big| \frac{*}{0}$ is contained in a perfect matching of $\frac{2}{*} \Big| \frac{*}{1}$, but by assumption $\sigma \notin \frac{2}{*} \Big| \frac{*}{1}$ so there are no unpaired faces.

Subcase 2: Suppose now that $\sigma \in \frac{2}{*} \Big| \frac{*}{*}$, $\sigma \in \frac{1}{*} \Big| \frac{*}{*}$, $\sigma \notin \frac{2}{2} \Big| \frac{2}{2}$, $\sigma \notin \frac{1}{1} \Big| \frac{1}{1}$, and $\sigma \notin \frac{2}{*} \Big| \frac{*}{0}$. Then it follows that $\sigma \in \frac{2}{*} \Big| \frac{*}{1}$ and $\sigma \in \frac{1}{*} \Big| \frac{*}{0}$. The intersection of the perfect matchings of $\frac{2}{*} \Big| \frac{*}{1}$ and $\frac{1}{*} \Big| \frac{*}{0}$ is contained in the perfect matching $\frac{1}{1} \Big| \frac{1}{1}$. Since this is a contradiction to the assumption that $\sigma \notin \frac{1}{1} \Big| \frac{1}{1}$, there are no unpaired faces left from this case.

Case 3.2: Suppose θ is of type (t2.1) and that $\delta \in \theta$. That is, $\theta = \sigma \cup \delta \cup \beta$ where $\alpha, \beta, \gamma, \delta \notin \sigma$ and $\sigma \cup \delta \in \frac{2}{*} \Big| \frac{*}{*}$, $\sigma \cup \delta \in \frac{1}{*} \Big| \frac{*}{*}$, $\sigma \cup \delta \notin \frac{2}{2} \Big| \frac{2}{2}$, and $(\sigma \cup \delta \notin \frac{2}{*} \Big| \frac{*}{1}$ or $\sigma \cup \delta \notin \frac{1}{1} \Big| \frac{1}{1})$. If $\theta \setminus \delta = \sigma \cup \beta$ is also of type (t2.1), then θ and $\theta \setminus \delta$ are paired

using δ . So assume $\sigma \cup \beta$ is not of type (t2.1). Therefore, $\sigma \notin \frac{2}{*} \left| \frac{*}{*} \right.$, or $\sigma \notin \frac{1}{*} \left| \frac{*}{*} \right.$, or $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$, or $(\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.)$. Notice that it is not possible for $\sigma \notin \frac{2}{*} \left| \frac{*}{*} \right.$ since $\sigma \cup \delta \in \frac{2}{*} \left| \frac{*}{*} \right.$ and, similarly, it is not possible for $\sigma \notin \frac{1}{*} \left| \frac{*}{*} \right.$ since $\sigma \cup \delta \in \frac{1}{*} \left| \frac{*}{*} \right.$. Hence, the faces left unpaired are such that $\sigma \in \frac{2}{*} \left| \frac{*}{0} \right.$, and $\sigma \in \frac{1}{*} \left| \frac{*}{0} \right.$, and $(\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$ or $(\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.)$). There are two subcases to consider:

Subcase 1: Suppose first that $\sigma \in \frac{2}{2} \left| \frac{2}{2} \right.$. Then, σ is in the intersection between the perfect matchings of $\frac{2}{*} \left| \frac{*}{0} \right.$, $\frac{1}{*} \left| \frac{*}{0} \right.$, and $\frac{2}{2} \left| \frac{2}{2} \right.$. Therefore the unpaired faces are $\sigma \cup \beta \cup \delta$ where $\sigma \in \{\emptyset, \{c_{0,0}\}, \{c_{1,1}\}, \{b_{3,2}\}, \{b_{2,2}\}, \{c_{0,0}, b_{3,2}\}, \{c_{0,0}, b_{2,2}\}, \{c_{1,1}, b_{3,2}\}, \{c_{1,1}, b_{2,2}\}\}$.

Subcase 2: Suppose now that $\sigma \in \frac{2}{*} \left| \frac{*}{1} \right.$ and $\sigma \in \frac{1}{1} \left| \frac{1}{1} \right.$. The intersection of $\frac{2}{*} \left| \frac{*}{0} \right.$, $\frac{1}{*} \left| \frac{*}{0} \right.$, $\frac{2}{*} \left| \frac{*}{1} \right.$, and $\frac{1}{1} \left| \frac{1}{1} \right.$ is $\mathcal{P}(b_{0,2}, a_{1,1}, c_{0,0}, c_{3,3}, a_{3,2}, b_{3,2})$.

Therefore, the unpaired faces obtained from Case 3.2 are those of the form $\sigma \cup \beta \cup \delta$ where $\sigma \in \mathcal{P}(b_{0,2}, a_{1,1}, c_{0,0}, c_{3,3}, a_{3,2}, b_{3,2}) \cup \{\{c_{1,1}\}, \{b_{2,2}\}, \{c_{0,0}, b_{2,2}\}, \{c_{1,1}, b_{3,2}\}, \{c_{1,1}, b_{2,2}\}\}$.

We are now left with three types of unpaired faces:

- (1) $\sigma \cup \beta \cup \gamma$ where $\sigma \in \mathcal{P}(c_{0,0}, b_{3,2}, c_{1,0}, b_{2,1})$ (Case 4.2)
- (2) $\sigma \cup \gamma \cup \delta$ where $\sigma \in \mathcal{P}(b_{0,2}, b_{1,3}, c_{3,3}, c_{2,3}) \cup \{\{b_{1,2}\}, \{c_{2,2}\}, \{b_{1,2}, c_{2,2}\}, \{b_{1,2}, c_{3,3}\}, \{b_{0,2}, c_{2,2}\}\}$ (Case 2.2)
- (3) $\sigma \cup \beta \cup \delta$ where $\sigma \in \mathcal{P}(b_{0,2}, a_{1,1}, c_{0,0}, c_{3,3}, a_{3,2}, b_{3,2}) \cup \{\{c_{1,1}\}, \{b_{2,2}\}, \{c_{0,0}, b_{2,2}\}, \{c_{1,1}, b_{3,2}\}, \{c_{1,1}, b_{2,2}\}\}$. (Case 3.2)

Pairing with $c_{1,0}$ matches all faces in (1). Then, pairing with $b_{0,2}$ leaves the faces:

- (2) $\sigma \cup \gamma \cup \delta$ where $\sigma \in \{\{b_{1,2}\}, \{b_{1,2}, c_{2,2}\}, \{b_{1,2}, c_{3,3}\}\}$
- (3) $\sigma \cup \beta \cup \delta$ where $\sigma \in \{\{c_{1,1}\}, \{b_{2,2}\}, \{c_{0,0}, b_{2,2}\}, \{c_{1,1}, b_{3,2}\}, \{c_{1,1}, b_{2,2}\}\}$.

Finally we perform a small series of element pairings. The element pairing using $c_{3,3}$ pairs $\{b_{1,2}\}$ with $\{b_{1,2}, c_{3,3}\}$, an element pairing using $c_{0,0}$ pairs $\{b_{2,2}\}$ with $\{c_{0,0}, b_{2,2}\}$, and an element pairing using $b_{3,2}$ pairs $c_{1,1}$ with $\{c_{1,1}, b_{3,2}\}$. This leaves us with two critical cells $\{b_{1,2}, c_{2,2}\} \cup \delta \cup \gamma$ and $\{c_{1,1}, b_{2,2}\} \cup \beta \cup \delta$ and the homotopy type $S^3 \vee S^3$. \square

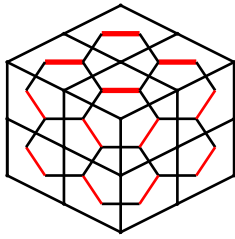
5 Conclusion and further directions

Throughout this paper, our main guiding question has been, are the perfect matching complexes of honeycomb graphs all contractible or homotopy equivalent to a wedge of spheres? We have considered the homotopy type of the perfect matching complexes of honeycomb graphs $H_{\ell \times m \times n}$. When $\ell = 1$, we were able to compute all homotopy types, but for $\ell \geq 2$, the only homotopy type we were able to compute is for $H_{2 \times 2 \times 2}$. It appears, at this time, that for larger honeycomb graphs, we need a new strategy.

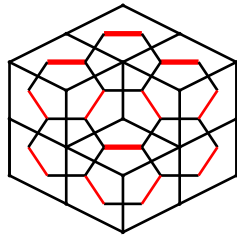
Our motivation for this project has been to better understand (ordinary) matching complexes of honeycomb graphs, $\mathcal{M}(H_{k \times m \times n})$. Since not every maximal matching of a honeycomb graph is contained in a perfect matching, the matching complex is not a pure complex, that is, not all maximal faces of the matching complex are the same dimension. The perfect matching complex is a pure complex, and thus less complicated than the matching complex. For example, Matsushita ([21]) showed that the matching complex of the honeycomb graphs $H_{1,1,n}$ has the homotopy type of a wedge of spheres, as part of a more general result on polygonal line tilings. This contrasts with our result on these graphs: if you consider only the subcomplex of perfect matchings, the complex is contractible (for $n \geq 2$).

We are interested more generally in the relationship between the (ordinary) matching complex and the perfect matching complex. Our curiosity in (ordinary) matching complexes has not diminished. Are the matching complexes of honeycomb graphs contractible or homotopy equivalent to wedges of spheres? What can we say about the (ordinary or perfect) matching complexes of more general classes of graphs, such as bipartite graphs?

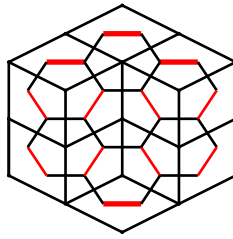
A Perfect matchings of $H_{2 \times 2 \times 2}$



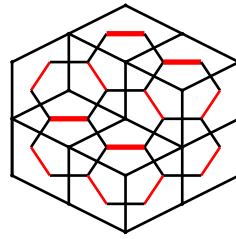
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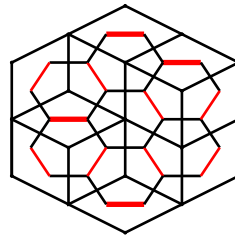
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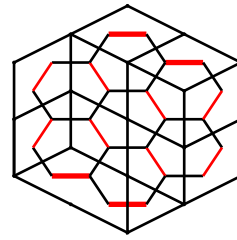
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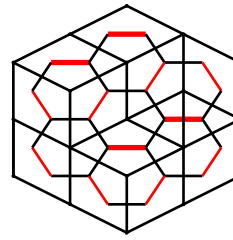
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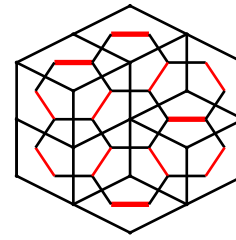
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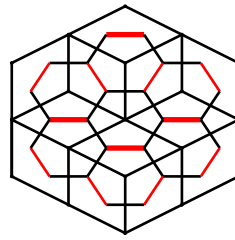
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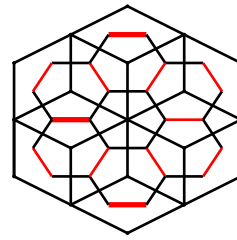
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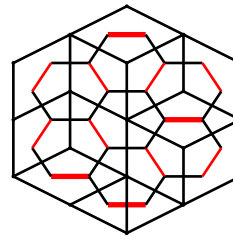
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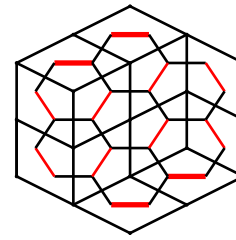
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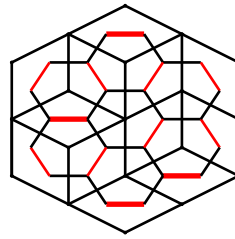
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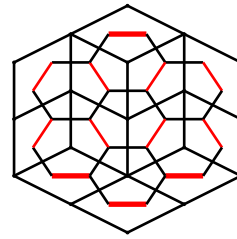
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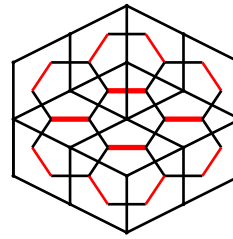
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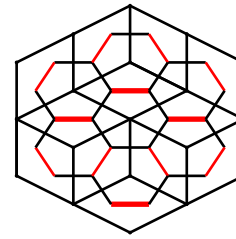
$$\frac{2 \mid 0}{1 \mid 0}$$



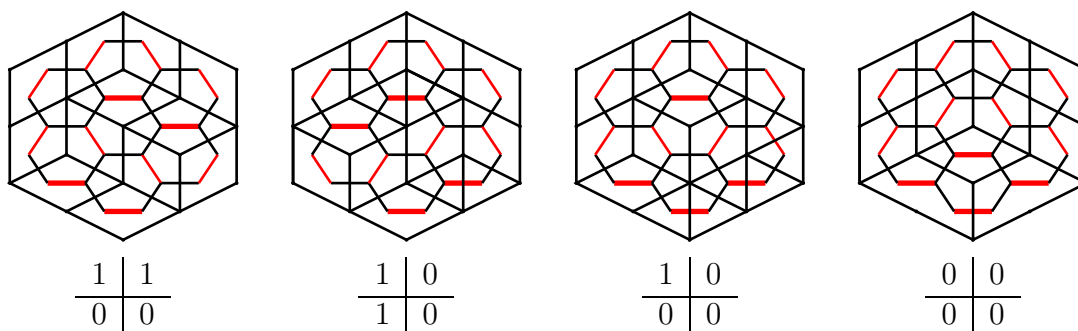
$$\frac{2 \mid 0}{0 \mid 0}$$



$$\frac{1 \mid 1}{1 \mid 1}$$



$$\frac{1 \mid 1}{1 \mid 0}$$



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