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SPECTRA OF SUBDIVISIONS OF SIGNED GRAPHS, SIGNED R-GRAPHS AND RELATED PRODUCTS

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Abstract. The subdivision is a bipartite graph built from an ordinary graph by inserting a vertex into every edge, and an R-graph is obtained by adding a new vertex to every edge and joining it to the ends of the corresponding edge. In this paper we deal with similar constructions for signed graphs. Both are stable under switching, and the question on balance is completely resolved. In the regular case, the spectrum of the adjacency matrix of signed R-graph is computed. We also introduce two corona-like products based on the subdivision of a signed graph and four similar products based on the signed R-graph operation. For each of them we compute the characteristic polynomial along with the spectrum of the adjacency matrix and the spectrum of the Laplacian matrix either in general case or in case when one constituent is just regular or simultaneously regular and net-regular. In addition, we consider an other operation, called the generalized subdivision, introduced in [Ars Math. Contemp. **23** (2023), 3–9] and compute the spectrum of its adjacency matrix in terms of the Laplacian spectrum of the corresponding signed graph. In this way, we positively address a problem posed in the same reference. Our results can be interesting in the context of signed graphs sharing the same spectrum, since they provide constructions of such signed graphs in case of the ordinary spectrum as well as in case of the Laplacian spectrum.

1. Introduction

A *signed graph* Σ is a pair (G, σ) where $G = (V, E)$ is an unoriented connected graph without loops or multiple edges, also known as the *underlying graph*, and $\sigma\colon E(G) \longrightarrow \{1, -1\}$ is the *sign function* (or the *signature*). We interchangeably use $V(G)$ and $V(\Sigma)$ to denote the vertex set of Σ. The edge set of a signed graph is composed of positive and negative edges. An ordinary (unsigned) graph is interpreted as a signed graph with all-positive signature.

The *adjacency matrix* $A(\Sigma) = (a_{ij})$ is obtained from the standard adjacency matrix of the underlying graph G by reversing the sign of every entry that corresponds to a negative edge. The *characteristic polynomial* and *eigenvalues* of Σ are

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the characteristic polynomial and the eigenvalues of $A(\Sigma)$. The eigenvalues form the *spectrum* of Σ. The *Laplacian matrix* of Σ is $L(\Sigma) = D_G - A(\Sigma)$ where D_G is the diagonal matrix of vertex degrees of G. The *Laplacian characteristic polynomial*, the *Laplacian eigenvalues* and the *Laplacian spectrum* of Σ correspond to the same objects related to $L(\Sigma)$. We use ϕ_Σ and $\lambda_1, \lambda_2, \dots, \lambda_n$ (resp. $\psi_\Sigma(x)$ and $\mu_1, \mu_2, \dots, \mu_n$) to denote the characteristic polynomial and the eigenvalues (Laplacian characteristic polynomial and the Laplacian eigenvalues) of Σ, respectively. It is assumed that the eigenvalues are indexed non-increasingly.

Many notions about unsigned graphs extend directly to signed graphs. For example, a signed graph is connected, regular or bipartite, if the same holds for its underlying graph. The vertex degree in a signed graph is its degree in the underlying graph. However, some notions are reserved for signed graphs. The *net degree* $d^{\pm}_{\Sigma}(v_i)$ of a vertex v is the difference between the number of positive and the number of negative edges incident with v in Σ. We say that Σ is *s-net-regular* if $d^{\pm}_{\Sigma}(v) = s$, for all $v \in V(\Sigma)$. Similarly, Σ is *co-regular* if the underlying graph is r-regular and Σ is s-net-regular [8]. (We note in passing that a net-regular signed graph does not need to be regular.) A class of such signed graphs is denoted by (r, s). The sign of a cycle is the product of signs of its edges, and a signed cycle is *positive* (resp. *negative*) if its sign is 1 (resp. -1).

A fundamental concept in the theory of signed graphs is the switching equivalence: If U is a subset of the vertex set of Σ, let Σ^U denote the signed graph obtained by reversing the sign of every edge with one end in U and the other in $V(\Sigma) \setminus U$. Then Σ^U is switching equivalent to Σ. We use to say that Σ and Σ^U switch to each other. In matrix terminology, the signed graphs Σ and Σ' are switching equivalent if there exists a diagonal matrix X with ± 1 on the main diagonal such that $A(\Sigma') = X^{-1}A(\Sigma)X$. It follows that switching is an equivalence relation that preserves the spectrum of the adjacency matrix and the spectrum of the Laplacian matrix. We say that two signed graphs are switching isomorphic if one of them is isomorphic to a switching of the other one. Two signed graphs are cospectral (resp. Laplacian cospectral) if they are not switching isomorphic but share the same spectrum (Laplacian spectrum).

An other fundamental concept is balance: A signed graph or its subgraph is *balanced* if every cycle in it, if any, is positive; otherwise, it is *unbalanced.* It is not difficult to see that each cycle in Σ maintains its sign after switching. Consequently, a signed graph is balanced if and only if it switches to its underlying graph [17]. A simple spectral criterion states that a signed graph is balanced if and only if its largest eigenvalue (resp. least Laplacian eigenvalue) is equal to the largest eigenvalue of its underlying graph (equal to zero) [14, 18].

There are many operations on ordinary graphs, and for a survey we refer the reader to [1, 2, 5]. Some of them are transferred to signed graphs; for example, there are two parallel definitions of a signed line graph [4, 18], and there is a definition of a total graph of a signed graph [4] (not quoted in this paper).

In this paper we are interested in the following two graph operations: The subdivision $S(G)$ of an ordinary graph G is the bipartite graph obtained by inserting

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a new vertex onto every edge of G. Similarly, the R-graph is obtained by adding a new vertex corresponding to each edge of G and joining each new vertex to the ends of the corresponding edge [5]. The authors of [3] introduced the notion of a subdivision of a signed graph, and in this paper we do the same for the signed R-graph (for short, the SR-graph). These definitions require additional notions, and thus they are given later in Section 3. Neither of them generalizes the corresponding definition of an ordinary graph (and the same holds for the foregoing signed line graph and total graphs of signed graphs). The latter is expected due to the nature of signed graphs, and a more significant is the fact that all definitions are stable under the switching equivalence (in the sense that switching equivalent signed graphs produce switching equivalent subdivisions or SR-graphs), and that the question on balance of the resulting signed graph is completely resolved. In other words, the definitions nicely fit into both aforementioned fundamental concepts. With certain modifications, the same definitions can be generalized to graphs posing orientations, multiple edges or self-loops. However, the idea in this paper is to establish the concept in the basic environment.

The spectrum of the subdivision of a signed graph is computed in [3], and here we compute the spectrum of the SR-graph of a regular signed graph. We also introduce two signed graph products defined on the basis of subdivisions, as well as four products defined on the basis of SR-graphs. All products are corona-like, in the sense that they include a fixed copy of a signed graph, say Σ, and a set of disjoint copies of an other signed graph where this set is in bijection with the set of either vertices or edges of Σ. These two signed graphs are referred to as the constituents of the corresponding product. For definition of corona product of ordinary graphs, see [1]. We compute the spectral parameters (the characteristic polynomial, the spectrum, the Laplacian characteristic polynomial spectrum and the Laplacian spectrum) for each of them either in general case or in case in which one of constituents is regular or co-regular. We also consider an other subdivision-based operation introduced in [13], and compute its spectrum in terms of the Laplacian spectrum of the corresponding signed graph. In this way, we positively address a research problem posed in the same reference.

Our results can be interesting in the context of cospectral or Laplacian cospectral signed graphs, as they provide constructions of such signed graphs. Concerning related results, they are often meet in the framework of ordinary graphs; see [1] (for spectra of many graph products) [5] (for a distinguishable number of classical results concerning graph compositions and their spectra), [11] (for spectra of graph products based on R-graphs) or [2, 7, 10, 12] (for spectra of corona-like products of graphs). Some products of signed graphs and their spectra can be found in [6, 13].

Here is the content of the remaining sections. Section 2 is preparatory. Definitions of a subdivision and an SR-graph are given in Section 3. This section also contains a discussion on balance and the proof that the SR-graph operation is stable under the switching equivalence. The spectrum in the regular case is also established. Products based on the subdivision are considered in Section 4. In this

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section we also address the question of [13]. Four products based on the SR-graph operation have received the attention in Section 5.

2. Preliminaries

We use $\mathbf{j}$, O, J and I to denote the all-1 vector, the all-0 matrix, the all-1 matrix and the identity matrix, respectively. The size may be given in the subscript. The direct sum of the matrices M and N is denoted by $M \oplus N$. Throughout the paper, we use the exponential notation for the spectrum (or the Laplacian spectrum) in which an exponent stands for the multiplicity of the corresponding eigenvalue.

We believe that the reader is familiar with the Kronecker product, but for the sake of completeness and to recall some properties of this product, we give the following details. For an $m \times n$ matrix P and a $p \times q$ matrix Q, the *Kronecker product* $P \otimes Q$ is the $mp \times nq$ matrix obtained from P by replacing each element p_{ij} by $p_{ij}Q$ [5, 9]. This is an associative operation with the property that $(P \otimes Q)^\intercal = P^\intercal \otimes Q^\intercal$ and $(P \otimes Q)(R \otimes S) = PR \otimes QS$ whenever the products PR and QS exist. The later implies $(P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}$ whenever P and Q are invertible. Moreover, if $m = n$ and $q = p$, then $\det(P \otimes Q) = (\det P)^p (\det Q)^n$. These properties will be used in the forthcoming sections without noting.

Following [9], we say that the *M-coronal* $\chi_M(x)$ of an $n \times n$ matrix M is the entry sum of $(xI_n - M)^{-1}$, that is $\chi_M(x) = \mathbf{j}^\intercal (xI_n - M)^{-1}\mathbf{j}$. If M has a constant row sum l, then $\chi_M(x) = \frac{n}{x-l}$.

3. Subdivisions and SR-graphs

To define the operations of the last title, we need the following notions. In the spirit of [4, 18], for a signed graph $\Sigma = (G, \sigma)$, we introduce the *vertex-edge orientation* $\eta\colon V(G) \times E(G) \longrightarrow \{1, 0, -1\}$ formed by obeying these rules: (a) $\eta(u, vw) = 0$ whenever $u \notin \{v, w\}$, (b) $\eta(v, vw) = 1$ or $\eta(v, vw) = -1$ and (c) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$. The *incidence matrix* $B_\eta = (\eta_{ij})$ is a vertex-edge incidence matrix derived from Σ, such that its (i, j)-entry is $\eta(i, j)$. This matrix plays a significant role in defining several products on signed graphs, see [4, 17]. In addition, the Laplacian matrix is obtained as a row-by-row product of the matrix B_η with itself. Note that the eigenvalues of $B_\eta^\intercal B_\eta$ are the eigenvalues of $B_\eta B_\eta^\intercal$ together with 0 of multiplicity $m - n$, where $|V(G)| = n$ and $|E(G)| = m$.

We proceed with the *subdivision* of a signed graph. Let Σ_η be a signed graph Σ accompanied with the vertex-edge orientation η. The adjacency matrix of the subdivision of Σ_η is

$$A(S(\Sigma_\eta)) = \begin{pmatrix} O_n & B_\eta \\ B_\eta^\intercal & O_m \end{pmatrix}. \tag{3.1}$$

Evidently, $S(\Sigma_\eta)$ is bipartite. An example is illustrated in Figure 1. In the two-steps procedure, a vertex-edge orientation is assigned according to defining rules (a)–(c), and the subdivision is constructed according to the previous definition (i.e., it is extracted from the adjacency matrix (3.1)). Further details are given in the next remark.

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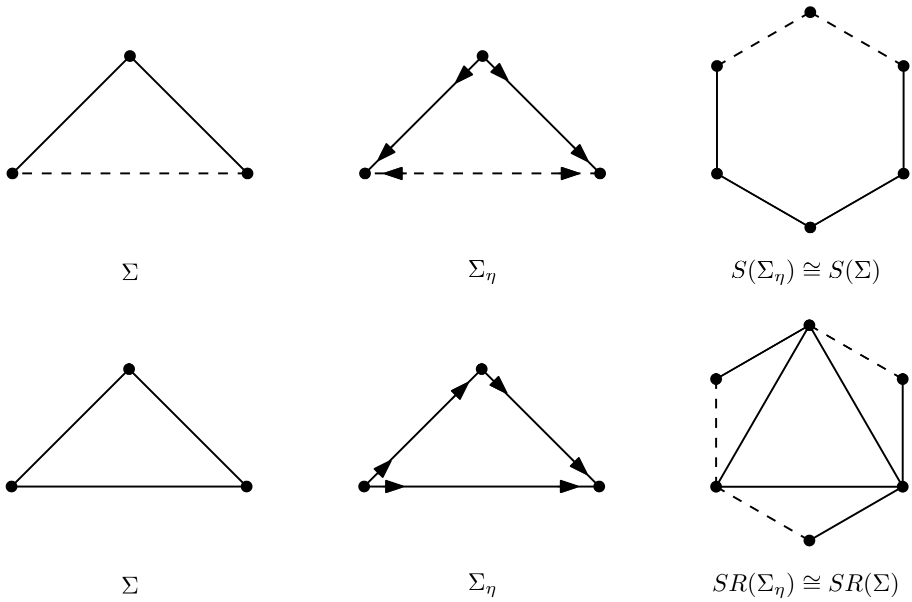

FIGURE 1. Signed graphs, vertex-edge orientations, and the corresponding subdivisions and SR-graphs. Negative edges are dashed.

Remark 3.1. Any vertex-edge orientation η gives rise to the same adjacency matrix and the same Laplacian matrix of Σ. However, $A(S(\Sigma_\eta))$ depends on η. Let $B_{\eta'} = B_\eta S$, where S is a diagonal matrix with ± 1 on the main diagonal. It can be easily seen that $A(S(\Sigma_{\eta'})) = (I_n \oplus S)A(S(\Sigma_\eta))(I_n \oplus S)$. In other words, changing the vertex-edge orientation results in a switching equivalent subdivision.

We have noted in the introductory section that the subdivision of a signed graph has attracted an attention in recent literature. Accordingly, we quote the following result concerning the spectrum.

Lemma 3.2 ([3])**.** *Let Σ be a signed graph with n vertices and m edges. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ are its Laplacian eigenvalues, then the eigenvalues of (the adjacency matrix of) $S(\Sigma)$ are $\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \ldots, \pm\sqrt{\mu_{n-1}}$ and 0 with multiplicity $m-n+2$ if Σ is balanced, and $\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \ldots, \pm\sqrt{\mu_{n-1}}, \pm\sqrt{\mu_n}$ and 0 with multiplicity $m-n$, otherwise.*

We continue with *signed R-graphs* (or *SR-graphs*, as we already said). The adjacency matrix of the SR-graph of a signed graph Σ_η is

$$A(SR(\Sigma_\eta)) = \begin{pmatrix} A(\Sigma) & B_\eta \\ B_\eta^\intercal & O_m \end{pmatrix}.$$

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It is easy to see that the signature σ_{SR} is defined by $\sigma_{SR}(vw) = \sigma(vw)$ and $\sigma_{SR}(uv) = \eta(v, vw)$, where σ is the signature of Σ and u is a new vertex corresponding to the edge vw. For an example, see again Figure 1. There, an unsigned graph appears in the role of Σ as it is interpreted as a signed graph with without negative edges.

Remark 3.3. Needless to add, the underlying graph of $S(\Sigma_\eta)$ is the subdivision of the underlying graph G. Similarly, $SR(\Sigma_\eta)$ is underlined by $R(G)$. One may also observe that there is an other way to define the subdivision or the SR-graph: In the first case every edge falls apart into two new edges that inherit the sign from the original one, and in the second case two edges incident with a new vertex, say u, share the sign with the edge that corresponds to u. However, our concept does not deviate from definitions of a signed line graph and a total graph of a signed graph [4, 18].

Finally, considering an ordinary graph as a signed graph with the all-positive signature, we observe that definitions of a subdivision and an SR-graph do not generalize the corresponding definitions in the context of graphs.

We resolve the question on balance of $S(\Sigma)$ and $SR(\Sigma)$. It follows from definition of $S(\Sigma)$ that this signed graph is balanced if and only if every cycle in Σ has an even number of positive edges; in other words, every even cycle is positive and every odd cycle is negative.

If Σ has $t = t^+ + t^-$ triangles (where t^+ and t^- denote the number of positive and the number negative triangles, respectively), then $SR(\Sigma_\eta)$ has exactly t^+ positive and $t^- + m$ negative triangles. Consequently, an SR-graph with at least one edge is always unbalanced.

The remainder of this section is exclusive to SR-graphs. We show that $SR(\Sigma)$ is stable under changing the vertex-edge orientation and under switching on Σ. We also compute the eigenvalues of $SR(\Sigma)$ in terms of eigenvalues of Σ, when Σ is regular.

Theorem 3.4. *Let η_1 and η_2 be two vertex-edge orientations on Σ. Then $SR(\Sigma_{\eta_1})$ and $SR(\Sigma_{\eta_2})$ are switching equivalent.*

Proof. There exists an $m \times m$ diagonal matrix X with ± 1 on the main diagonal such that $B_{\eta_1} = B_{\eta_2} X$. Now,

$$\begin{aligned} A(SR(\Sigma_{\eta_1})) &= \begin{pmatrix} A(\Sigma) & B_{\eta_1} \\ B_{\eta_1}^\intercal & O \end{pmatrix} = \begin{pmatrix} A(\Sigma) & B_{\eta_2}X \\ X^\intercal B_{\eta_2}^\intercal & O \end{pmatrix} \\ &= \begin{pmatrix} I & O \\ O & X^\intercal \end{pmatrix} \begin{pmatrix} A(\Sigma) & B_{\eta_2} \\ B_{\eta_2}^\intercal & O \end{pmatrix} \begin{pmatrix} I & O \\ O & X \end{pmatrix} = \left(I \oplus X\right) A(SR(\Sigma_{\eta_2})) \left(I \oplus X\right), \end{aligned}$$

and we are done, since the adjacency matrices are switching similar. ☐

Henceforth, the subscript η will not be specified.

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Theorem 3.5. *If Σ_1 and Σ_2 are switching equivalent signed graphs, then $SR(\Sigma_1)$ and $SR(\Sigma_2)$ are also switching equivalent.*

Proof. It holds $A(\Sigma_1) = X^{-1}A(\Sigma_2)X$, for some switching matrix X. If B is a vertex-edge incidence matrix of Σ_1, then $B' = XB$ plays the same role for Σ_2. Now,

$$\begin{aligned} A(SR(\Sigma_2)) &= \begin{pmatrix} A(\Sigma_2) & B' \\ B'^{\intercal} & O \end{pmatrix} = \begin{pmatrix} XA(\Sigma_1)X^{-1} & XB \\ (XB)^{\intercal} & O \end{pmatrix} \\ &= \begin{pmatrix} X & O \\ O & I \end{pmatrix} \begin{pmatrix} A(\Sigma_1) & B \\ B^{\intercal} & O \end{pmatrix} \begin{pmatrix} X^{-1} & O \\ O & I \end{pmatrix} = \left(X \oplus I\right) A(SR(\Sigma_1)) \left(X \oplus I\right), \end{aligned}$$

which leads to the desired result. □

We compute the spectrum of $SR(\Sigma)$ when Σ is regular.

Theorem 3.6. *Let Σ be an r-regular signed graph with n vertices and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The eigenvalues of $SR(\Sigma)$ are 0 with multiplicity $(\frac{r}{2} - 1)n$ and $\frac{1}{2}(\lambda_i \pm \sqrt{\lambda_i{}^2 - 4(\lambda_i - r)})$, for $1 \le i \le n$.*

Proof. Since Σ is r-regular, we have

$$BB^{\intercal} = L(\Sigma) = D(G) - A(\Sigma) = rI - A(\Sigma),$$

for a vertex-edge incidence matrix B. Using the previous identity and the Schur complement formula (see [5, Lemma 2.2]), we compute the characteristic polynomial of $SR(\Sigma)$ as follows

$$\begin{aligned} \phi_{SR(\Sigma)}(x) &= \det \begin{pmatrix} xI_n - A(\Sigma) & -B \\ -B^{\intercal} & xI_m \end{pmatrix} = \det(xI_m)\det(xI - A(\Sigma) - B(xI_m)^{-1}B^{\intercal}) \\ &= x^m \det\left(xI - A(\Sigma) - \frac{BB^{\intercal}}{x}\right) = x^{(\frac{r}{2}-1)\cdot n} \prod_{i=1}^{n} (x^2 - \lambda_i x - r + \lambda_i). \end{aligned}$$

The roots of $x^2 - \lambda_i x - r + \lambda_i$ are $\frac{1}{2}(\lambda_i \pm \sqrt{\lambda_i{}^2 - 4(\lambda_i - r)})$, and we are done. □

This section is concluded with a corollary concerning the distribution of the eigenvalues of $SR(\Sigma)$. We recall from the introductory section that λ_1 and λ_n denote the largest and the least eigenvalue, respectively.

Corollary 3.7. *Under the assumptions of Theorem 3.6, the spectrum of $SR(\Sigma)$ lies in $[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 - 4(\lambda_n - r)}), \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 - 4(\lambda_1 - r)})]$.*

Proof. For $x \in [-r, r]$, the function $f_1(x) = \frac{1}{2}(x + \sqrt{x^2 - 4(x - r)})$ is increasing. Hence, its maximum is attained at $x = \lambda_1$. Similarly, the function $f_2(x) = \frac{1}{2}(x - \sqrt{x^2 - 4(x - r)})$ is also increasing, and thus its minimum is attained at $x = \lambda_n$. Therefore, the entire spectrum of $SR(\Sigma)$ lies in $[f_2(\lambda_n), f_1(\lambda_1)]$, as desired. □

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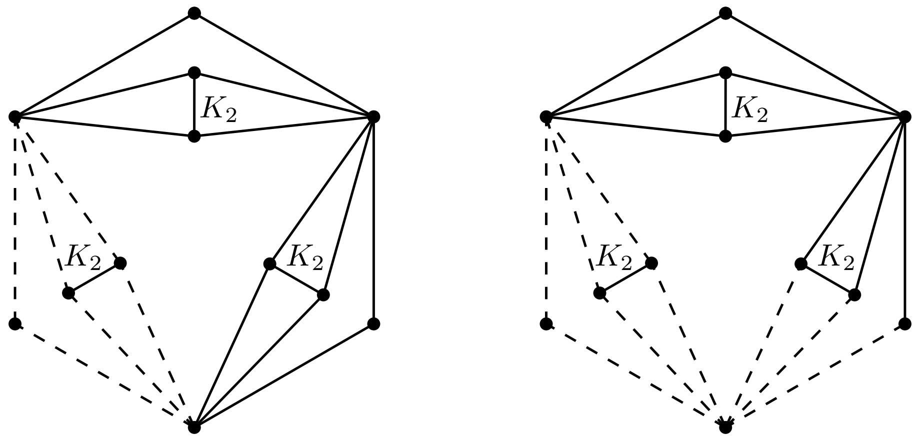

Figure 2. The subdivision-vertex neighbourhood corona of a negative triangle and K_2, and the subdivision-edge neighbourhood corona of the same constituents.

4. Products based on subdivisions and the corresponding spectra

In this section, we define two products of signed graphs based on subdivisions and compute the characteristic polynomials of the adjacency matrix. Under certain regularity assumptions, we deal with the Laplacian characteristic polynomial and eigenvalues of both matrices. We also consider the generalized subdivision (of a signed graph) introduced in [13], and compute the corresponding spectrum. In this way, we positively answer a research problem posed in the same reference.

Throughout the section we deal with two signed graphs, Σ_1 and Σ_2, and assume that Σ_i has n_i vertices and m_i edges, for $i \in \{1, 2\}$.

Definition 4.1. The *subdivision-vertex neighbourhood corona* $\Sigma_1 \boxdot_S \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $S(\Sigma_1)$ and n_1 copies of Σ_2 by joining every neighbour, say v, of the vertex i of Σ_1 to every vertex in the ith copy of Σ_2 by an edge which inherits the sign from iv.

The signed graph $\Sigma_1 \boxdot_S \Sigma_2$ has $n_1+m_1+n_1n_2$ vertices and $2m_1+n_1m_2+2m_1n_2$ edges.

Definition 4.2. The *subdivision-edge neighbourhood corona* $\Sigma_1 \boxminus_S \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $S(\Sigma_1)$ and m_1 copies of Σ_2 by joining every neighbour, say v, of the vertex $i \in V(S(\Sigma_1)) \setminus V(\Sigma_1)$ to every vertex in the ith copy of Σ_2 by an edge which inherits the sign from iv.

The signed graph $\Sigma_1 \boxminus_S \Sigma_2$ has $n_1+m_1+m_1n_2$ vertices and $2m_1+m_1m_2+2m_1n_2$ edges. Figure 2 illustrates the previous products.

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With consistent vertex labellings, the adjacency matrices of $\Sigma_1 \boxdot_S \Sigma_2$ and $\Sigma_1 \boxminus_S \Sigma_2$ are

$$A(\Sigma_1 \boxdot_S \Sigma_2) = \begin{pmatrix} O & B & O \otimes J_{n_2}^{\intercal} \\ B^{\intercal} & O_{m_1 \times m_1} & B^{\intercal} \otimes J_{n_2}^{\intercal} \\ O \otimes J_{n_2} & B \otimes J_{n_2} & I_{n_1} \otimes A(\Sigma_2) \end{pmatrix}$$

and

$$A(\Sigma_1 \boxminus_S \Sigma_2) = \begin{pmatrix} O & B & B \otimes J_{n_2}^{\intercal} \\ B^{\intercal} & O_{m_1 \times m_1} & O \otimes J_{n_2}^{\intercal} \\ B^{\intercal} \otimes J_{n_2} & O \otimes J_{n_2} & I_{m_1} \otimes A(\Sigma_2) \end{pmatrix}.$$

If Σ_1 is an r-regular signed graph, then the Laplacian matrices of $\Sigma_1 \boxdot_S \Sigma_2$ and $\Sigma_1 \boxminus_S \Sigma_2$ are

$$L(\Sigma_1 \boxdot_S \Sigma_2) = \begin{pmatrix} rI_{n_1} & -B & O \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & (2+2n_2)I_{m_1} & -B^{\intercal} \otimes J_{n_2}^{\intercal} \\ O \otimes J_{n_2} & -B \otimes J_{n_2} & I_{n_1} \otimes (rI_{n_2} + L(\Sigma_2)) \end{pmatrix}$$

and

$$L(\Sigma_1 \boxminus_S \Sigma_2) = \begin{pmatrix} r(1+n_2)I_{n_1} & -B & -B \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & 2I_{m_1} & O \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} \otimes J_{n_2} & O \otimes J_{n_2} & I_{m_1} \otimes (2I_{n_2} + L(\Sigma_2)) \end{pmatrix}.$$

We compute the characteristic polynomial of $\Sigma_1 \boxdot_S \Sigma_2$.

Theorem 4.3. *Let Σ_1 be a signed graph with n_1 vertices, m_1 edges and Laplacian eigenvalues $\mu_1(\Sigma_1), \mu_2(\Sigma_1), \dots, \mu_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and eigenvalues $\lambda_1(\Sigma_2), \lambda_2(\Sigma_2), \dots, \lambda_{n_2}(\Sigma_2)$. Then the characteristic polynomial of $\Sigma_1 \boxdot_S \Sigma_2$ is*

$$\phi_{\Sigma_1 \boxdot_S \Sigma_2}(x) = x^{m_1} \prod_{i=1}^{n_2} (x - \lambda_i(\Sigma_2))^{n_1} \prod_{i=1}^{n_1} \Big(x - \Big(\chi_{A(\Sigma_2)}(x) + \frac{1}{x} \Big) \mu_i \Big).$$

Proof. We compute

$$\begin{aligned} \phi_{\Sigma_1 \boxdot_S \Sigma_2}(x) &= \det \begin{pmatrix} xI_{n_1} & -B & O \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & xI_{m_1} & -B^{\intercal} \otimes J_{n_2}^{\intercal} \\ O \otimes J_{n_2} & -B \otimes J_{n_2} & I_{n_1} \otimes (xI_{n_2} - A(\Sigma_2)) \end{pmatrix} \\ &= \det \begin{pmatrix} xI_{n_1} & -B & O \\ -B^{\intercal} & xI_{m_1} - \chi_{A(\Sigma_2)}(x) B^{\intercal} B & O \\ O \otimes J_{n_2} & -B \otimes J_{n_2} & I_{n_1} \otimes (xI_{n_2} - A(\Sigma_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (x - \lambda_i(\Sigma_2))^{n_1} \det(M), \end{aligned} \tag{4.1}$$

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where

$$\begin{aligned}
\det(M) &= \det\begin{pmatrix} xI_{n_1} & -B \\ -B^\intercal & xI_{m_1} - \chi_{A(\Sigma_2)}(x)B^\intercal B \end{pmatrix} \\
&= \det(xI_{n_1})\det\left(xI_{m_1} - \chi_{A(\Sigma_2)}(x)B^\intercal B - B^\intercal(xI_{n_1})^{-1}B\right) \\
&= x^{m_1}\prod_{i=1}^{n_1}\Big(x - \Big(\chi_{A(\Sigma_2)}(x) + \frac{1}{x}\Big)\mu_i(\Sigma_1)\Big). \qquad (4.2)
\end{aligned}$$

The desired result follows from (4.1) and (4.2). □

We say more about the spectrum of $\Sigma_1 \boxdot_S \Sigma_2$ when Σ_2 is net-regular. Observe that in this case, its net degree appears in its spectrum (and corresponds to the all-1 vector).

Theorem 4.4. *Suppose that, under the assumptions of Theorem 4.3, Σ_2 is s-net-regular and $\lambda_k(\Sigma_2) = s$, for some fixed k $(1 \leq k \leq n_2)$. The spectrum of $\Sigma_1 \boxdot_S \Sigma_2$ consists of*

(i) 0 *with multiplicity* $m_1 - n_1$,
(ii) $\lambda_i(\Sigma_2)$ *with multiplicity* n_1, *for* $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,
(iii) *and the roots of* $x^3 - sx^2 - (n_2+1)\mu_i(\Sigma_1)x + s\mu_i(\Sigma_1)$, *for* $1 \leq i \leq n_1$.

Proof. Since Σ_2 is s-net-regular, we have $\chi_{A(\Sigma_2)}(x) = \frac{n_2}{x-s}$. Now,

$$\begin{aligned}
x\Big(\chi_{A(\Sigma_2)}(x) + \frac{1}{x}\Big)\mu_i(\Sigma_1) &= x - \Big(\frac{n_2}{x-s} + \frac{1}{x}\Big)\mu_i(\Sigma_1) \\
&= \frac{1}{x(x-s)}\big(x^2(x-s) - (n_2x + x - s)\mu_i(\Sigma_1)\big) \\
&= \frac{1}{x(x-s)}\big(x^3 - sx^2 - (n_2+1)\mu_i(\Sigma_1)x + s\mu_i(\Sigma_1)\big).
\end{aligned}$$

The desired result follows from Theorem 4.3. □

Observe that if Σ_1 has l_1 $(1 \leq l_1 \leq n_1)$ distinct Laplacian eigenvalues and Σ_2 is an s-net-regular with l_2 $(1 \leq l_2 \leq n_2)$ distinct eigenvalues. Then, $\Sigma_1 \boxdot_S \Sigma_2$ has at most $3l_1 + l_2 + 1$ distinct eigenvalues.

Remark 4.5. Let Σ_1 and Σ_1' be Laplacian cospectral signed graphs, and Σ_2 any signed graph. Then, $\Sigma_1 \boxdot_S \Sigma_2$ and $\Sigma_1' \boxdot_S \Sigma_2$ cospectral.

If Σ_1 is any signed graph, and Σ_2 and Σ_2' are cospectral signed graphs with $\chi_{A(\Sigma_2)}(x) = \chi_{A(\Sigma_2')}(x)$, then $\Sigma_1 \boxdot_S \Sigma_2$ and $\Sigma_1 \boxdot_S \Sigma_2'$ cospectral.

If Σ_1 and Σ_1' are Laplacian cospectral signed graphs, and Σ_2 and Σ_2' are cospectral signed graphs with $\chi_{A(\Sigma_2)}(x) = \chi_{A(\Sigma_2')}(x)$, then $\Sigma_1 \boxdot_S \Sigma_2$ and $\Sigma_1' \boxdot_S \Sigma_2'$ are cospectral.

To arrive at cospectral signed graphs it is sufficient to select an appropriate pair of constituents according to the previous remark, and construct the corresponding subdivision-vertex neighbourhood corona. It is worth mentioning that every pair of regular graphs, say G and H, with the same number of vertices and the same vertex degree gives rise to a pair of cospectral regular signed graphs constructed

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in the following way: (1) insert a parallel negative edge between every pair of adjacent vertices of both graphs, (2) their signed line graphs are cospectral. This construction is obtained in [15, 16], and to the best of our knowledge, it has no counterpart in the domain of ordinary graphs. The corresponding signed graphs are regular, and therefore they are Laplacian cospectral, as well, and this is exactly what is required in the previous remark. In the forthcoming sections, we will meet similar discussions on cospectral or Laplacian cospectral products, and for constructions of particular examples, we refer to the previous method.

The proofs of the following three results are analogous to the proofs of the previous statements, so they are left to the reader. In the first, we assume that Σ_1 is regular and compute the Laplacian characteristic polynomial of $\Sigma_1 \boxdot_S \Sigma_2$.

Theorem 4.6. *Let Σ_1 be an r-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \ldots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and Laplacian eigenvalues $\mu_1(\Sigma_2), \mu_2(\Sigma_2), \ldots, \mu_{n_2}(\Sigma_2)$. The Laplacian characteristic polynomial of $\Sigma_1 \boxdot_S \Sigma_2$ is*

$$\begin{aligned}\psi_{\Sigma_1 \boxdot_S \Sigma_2}(x) =&(x-2-2n_2)^{m_1-n_1}\prod_{i=1}^{n_2}(x-r-\mu_i(\Sigma_2))^{n_1}\\ &\cdot\prod_{i=1}^{n_1}\Big((x-2-2n_2)(x-r)-(r-\lambda_i(\Sigma_1))(1+(x-r)\chi_{L(\Sigma_2)}(x-r))\Big).\end{aligned}$$

We proceed with the result analogous to Theorem 4.3.

Theorem 4.7. *Let Σ_1 be a signed graph with n_1 vertices, m_1 edges and Laplacian eigenvalues $\mu_1(\Sigma_1), \mu_2(\Sigma_1), \ldots, \mu_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and eigenvalues $\lambda_1(\Sigma_2), \lambda_2(\Sigma_2), \ldots, \lambda_{n_2}(\Sigma_2)$. The characteristic polynomial of $\Sigma_1 \boxminus_S \Sigma_2$ is*

$$\phi_{\Sigma_1 \boxminus_S \Sigma_2}(x) = x^{m_1}\prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{m_1}\prod_{i=1}^{n_1}\Big(x-\Big(\chi_{A(\Sigma_2)}(x)+\frac{1}{x}\Big)\mu_i(\Sigma_1)\Big).$$

Finally we compute the Laplacian characteristic polynomial of $\Sigma_1 \boxminus_S \Sigma_2$ when the first signed graph is regular.

Theorem 4.8. *Let Σ_1 be an r-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \ldots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and Laplacian eigenvalues $\mu_1(\Sigma_2), \mu_2(\Sigma_2), \ldots, \mu_{n_2}(\Sigma_2)$. The Laplacian characteristic polynomial of $\Sigma_1 \boxminus_S \Sigma_2$ is*

$$\begin{aligned}\psi_{\Sigma_1 \boxminus_S \Sigma_2}(x) =&(x-2)^{m_1-n_1}\prod_{i=1}^{n_2}(x-2-\mu_i(\Sigma_2))^{m_1}\\ &\cdot\prod_{i=1}^{n_1}\Big(x-r(1+n_2)-\Big(\frac{1}{x-2}+\chi_{L(\Sigma_2)}(x-2)\Big)\mu_i(\Sigma_1)\Big).\end{aligned}$$

We continue with a product introduced in [13].

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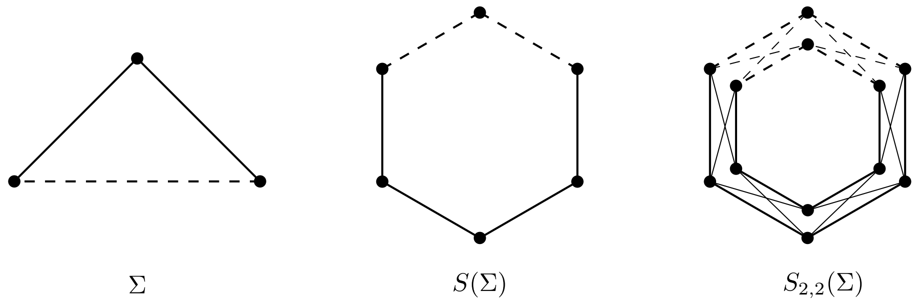

FIGURE 3. An example of the generalized subdivision.

Definition 4.9 ([13]). Let $S(\Sigma)$ be the subdivision of a signed graph Σ. The *generalized subdivision* $S_{k,p}(\Sigma)$ is obtained by replacing, in $S(\Sigma)$, every vertex of Σ by a cell of k non-adjacent vertices and every vertex of $V(S(\Sigma))\setminus V(\Sigma)$ by a cell of p non-adjacent vertices.

An example is illustrated in Figure 3. The spectrum of $S_{k,p}(\Sigma)$ is computed in [13] for $p\in\{k-1,k\}$. Here we proceed to establish the result for arbitrary values of p and k.

Theorem 4.10. *Let Σ be a signed graph with n vertices, m edges and Laplacian eigenvalues $\mu_1\geq\mu_2\geq\cdots\geq\mu_{n-1}>\mu_n\geq 0$. The eigenvalues of $S_{k,p}(\Sigma)$ are*

$$\begin{array}{ll} 0^{(k-1)n+pm+2}, \pm\sqrt{pk\mu_1}, \pm\sqrt{pk\mu_2}, \dots, \pm\sqrt{pk\mu_{n-1}} & \text{if } \Sigma \text{ is balanced,} \\ 0^{(k-1)n+pm}, \pm\sqrt{pk\mu_1}, \pm\sqrt{pk\mu_2} \dots, \pm\sqrt{pk\mu_{n-1}}, \pm\sqrt{pk\mu_n} & \text{if } \Sigma \text{ is unbalanced.} \end{array}$$

Proof. We distinguish the following cases.

Case 1: $k\leq p$. With consistent vertex labellings, we get

$$A(S_{k,p}(\Sigma))=\begin{pmatrix} O & B & O & B\cdots & O & B & B & B\cdots & B \\ B^\intercal & O & B^\intercal & O\cdots & B^\intercal & O & O & O\cdots & O \\ O & B & O & B\cdots & O & B & B & B\cdots & B \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots & \\ O & B & O & B\cdots & O & B & B & B\cdots & B \\ B^\intercal & O & B^\intercal & O\cdots & B^\intercal & O & O & O\cdots & O \\ B^\intercal & O & B^\intercal & O\cdots & B^\intercal & O & O & O\cdots & O \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots & \\ B^\intercal & O & B^\intercal & O\cdots & B^\intercal & O & O & O\cdots & O \end{pmatrix}.$$

Subcase 1.1: Σ is balanced. Let $\mathbf{x}$ and $\mathbf{y}$ be $n\times 1$ and $m\times 1$ vectors such that $\mathbf{z}=\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$ is an eigenvector corresponding to the non-zero eigenvalue λ_i, $1\leq i\leq 2n-2$, of $S(\Sigma)$. Then $A(S(\Sigma))\mathbf{z}=\lambda_i\mathbf{z}$ implies $B\mathbf{y}=\lambda_i\mathbf{x}$ and $B^\intercal\mathbf{x}=\lambda_i\mathbf{y}$. Consider the $(kn+pm)\times 1$ vector $\mathbf{w}=(\sqrt{p}\mathbf{x}^\intercal,\sqrt{k}\mathbf{y}^\intercal,\sqrt{p}\mathbf{x}^\intercal,\dots,\sqrt{p}\mathbf{x}^\intercal,\sqrt{k}\mathbf{y}^\intercal,\sqrt{k}\mathbf{y}^\intercal,\dots,\sqrt{k}\mathbf{y}^\intercal)^\intercal$.

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We have

$$A(S_{k,p}(\Sigma))\mathbf{w} = \begin{pmatrix} O & B & O & B & \cdots & O & B & B & B & \cdots & B \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \\ O & B & O & B & \cdots & O & B & B & B & \cdots & B \\ \vdots & \vdots & \vdots & \vdots & & & & & & \vdots & \\ O & B & O & B & \cdots & O & B & B & B & \cdots & B \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & & & & & & \vdots & \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \end{pmatrix} \begin{pmatrix} \sqrt{p}\mathbf{x} \\ \sqrt{k}\mathbf{y} \\ \sqrt{p}\mathbf{x} \\ \vdots \\ \sqrt{p}\mathbf{x} \\ \sqrt{k}\mathbf{y} \\ \sqrt{p}\mathbf{y} \\ \vdots \\ \sqrt{p}\mathbf{y} \end{pmatrix}$$

$= \sqrt{pk}\lambda_i \mathbf{w}$. Therefore, $\sqrt{pk}\lambda_i$ is an eigenvalue of $S_{k,p}(\Sigma)$. Now, the result follows by Lemma 3.2.

Subcase 1.2: Σ is unbalanced. This subcase is proved by a slight modification of the proof of the previous one.

Case 2: $k \geq p$. Here we have

$$A(S_{k,p}(\Sigma)) = \begin{pmatrix} O & B & O & B & \cdots & O & B & B & B & \cdots & B \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \\ O & B & O & B & \cdots & O & B & & B & \cdots & B \\ \vdots & \vdots & \vdots & \vdots & & & & & & \vdots & \\ O & B & O & B & \cdots & O & B & B & B & \cdots & B \\ B^{\intercal} & O & B^{\intercal} & O & \cdots & B^{\intercal} & O & O & O & \cdots & O \\ O & B & O & B & \cdots & O & B & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & & & & & & \vdots & \\ O & B & O & B & \cdots & O & B & O & O & \cdots & O \end{pmatrix},$$

As before, we proceed with two subcases, where in the first one we construct the eigenvector $\mathbf{w} = (\sqrt{p}\mathbf{x}^{\intercal}, \sqrt{k}\mathbf{y}^{\intercal}, \sqrt{p}\mathbf{x}^{\intercal}, \ldots, \sqrt{p}\mathbf{x}^{\intercal}, \sqrt{k}\mathbf{y}^{\intercal}, \sqrt{p}\mathbf{x}^{\intercal}, \ldots, \sqrt{p}\mathbf{x}^{\intercal})^{\intercal}$ corresponding to $\sqrt{kp}\lambda_i$, and the second subcase is very similar. $\square$

5. Products based on SR-graphs and the corresponding spectra

As in the previous section, we deal with signed graphs Σ_i with n_i vertices and m_i edges, for $i \in \{1, 2\}$. We first define the products, and then consider their characteristic polynomials and spectra.

Definition 5.1. The *SR-vertex corona* $\Sigma_1 \odot_{SR} \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $SR(\Sigma_1)$ and n_1 copies of Σ_2 by joining the ith vertex of Σ_1 by a positive edge to every vertex in the ith copy of Σ_2.

Definition 5.2. The *SR-edge corona* $\Sigma_1 \ominus_{SR} \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $SR(\Sigma_1)$ and m_1 copies of Σ_2 by joining the ith vertex of $V(SR(\Sigma_1)) \setminus V(\Sigma_1)$ by a positive edge to every vertex in the ith copy of Σ_2.

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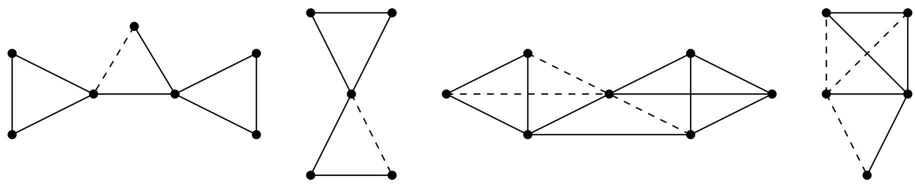

FIGURE 4. The SR-vertex corona, the SR-edge corona, the SR-vertex neighbourhood corona and the SR-edge neighbourhood corona of two copies of K_2.

Definition 5.3. The *SR-vertex neighbourhood corona* $\Sigma_1 \boxdot_{SR} \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $SR(\Sigma_1)$ and n_1 copies of Σ_2 by joining every neighbour, say v, of the vertex i of Σ_1 to every vertex in the ith copy of Σ_2 by an edge which inherits the sign from iv.

Definition 5.4. The *SR-edge neighbourhood corona* $\Sigma_1 \boxminus_{SR} \Sigma_2$ of Σ_1 and Σ_2 is the signed graph obtained from $SR(\Sigma_1)$ and m_1 copies of Σ_2 by joining every neighbour, say v, of the vertex $i \in V(SR(\Sigma_1)) \setminus V(\Sigma_1)$ to every vertex in the ith copy of Σ_2 by an edge that inherits the sign from iv.

Figure 4 illustrates the previous products. Each of them is separated in a subsection.

5.1. SR-vertex corona.

We first determine the characteristic polynomial of $\Sigma_1 \odot_{SR} \Sigma_2$ when Σ_1 is regular.

Theorem 5.5. *Let Σ_1 be an r_1-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \dots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and eigenvalues $\lambda_1(\Sigma_2), \lambda_2(\Sigma_2), \dots, \lambda_{n_2}(\Sigma_2)$. The characteristic polynomial of $\Sigma_1 \odot_{SR} \Sigma_2$ is*

$$\phi_{\Sigma_1 \odot_{SR} \Sigma_2}(x) = x^{m_1-n_1} \prod_{i=1}^{n_2} (x - \lambda_i(\Sigma_2))^{n_1} \prod_{i=1}^{n_1} \Big(x^2 - (\chi_{A(\Sigma_2)}(x) + \lambda_i(\Sigma_1))x - r_1 + \lambda_i(\Sigma_1) \Big).$$

Proof. If B be the incidence matrix of Σ_1, then with consistent vertex labellings, the adjacency matrix of $\Sigma_1 \odot_{SR} \Sigma_2$ is given by

$$A(\Sigma_1 \odot_{SR} \Sigma_2) = \begin{pmatrix} A(\Sigma_1) & B & I_{n_1} \otimes J_{n_2}^{\intercal} \\ B^{\intercal} & O_{m_1 \times m_1} & O_{m_1 \times n_1} \otimes J_{n_2}^{\intercal} \\ I_{n_1} \otimes J_{n_2} & O_{n_1 \times m_1} \otimes J_{n_2} & I_{n_1} \otimes A(\Sigma_2) \end{pmatrix}.$$

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We compute

$$\begin{aligned}
\phi_{\Sigma_1 \odot_{SR} \Sigma_2}(x) &= \det\begin{pmatrix} xI_{n_1} - A(\Sigma_1) & -B & -I_{n_1}\otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & xI_{m_1} & O_{m_1\times n_1}\otimes J_{n_2}^{\intercal} \\ -I_{n_1}\otimes J_{n_2} & O_{n_1\times m_1}\otimes J_{n_2} & I_{n_1}\otimes (xI_{n_2} - A(\Sigma_2))\end{pmatrix} \\
&= \det\begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)I_{n_1} & -B & O \\ -B^{\intercal} & xI_{m_1} & O \\ -I_{n_1}\otimes J_{n_2} & O_{n_1\times m_1}\otimes J_{n_2} & I_{n_1}\otimes (xI_{n_2} - A(\Sigma_2))\end{pmatrix} \\
&= \prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{n_1}\cdot \det(M), \qquad (5.1)
\end{aligned}$$

where $\det(M)$ is computed by the Schur complement formula as

$$\begin{aligned}
\det(M) &= \det\begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)I_{n_1} & -B \\ -B^{\intercal} & xI_{m_1}\end{pmatrix} \\
&= \det(xI_{m_1})\det\left(xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)I_{n_1} - B(xI_{m_1})^{-1}B^{\intercal}\right) \\
&= \det(xI_{m_1})\det\left(xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)I_{n_1} - \tfrac{1}{x}L(\Sigma_1)\right) \\
&= \det(xI_{m_1})\det\left(xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)I_{n_1} - \tfrac{1}{x}(r_1I_{n_1} - A(\Sigma_1))\right) \\
&= x^{m_1-n_1}\prod_{i=1}^{n_1}\Big(x^2 - (\chi_{A(\Sigma_2)}(x) + \lambda_i(\Sigma_1))x - r_1 + \lambda_i(\Sigma_1)\Big). \qquad (5.2)
\end{aligned}$$

Equalities (5.1) and (5.2) lead to the result. □

We can say more if we set Σ_2 to be net-regular.

Theorem 5.6. *Suppose that, under the assumptions of Theorem 5.5, Σ_2 is s-net-regular with $\lambda_k(\Sigma_2) = s$ for some fixed k ($1 \le k \le n_2$). The spectrum of $\Sigma_1 \odot_{SR} \Sigma_2$ consists of*

(i) 0 *with multiplicity* $m_1 - n_1$,
(ii) $\lambda_i(\Sigma_2)$ *with multiplicity* n_1, *for* $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,
(iii) *and the roots of* $x^3 - (s + \lambda_i(\Sigma_1))x^2 - (n_2 - (s+1)\lambda_i(\Sigma_1) + r_1)x + (r_1 - \lambda_i(\Sigma_1))s$, *for* $1 \le i \le n_1$.

Proof. Item (i) follows directly from Theorem 5.5.

Since Σ_2 is s-net-regular, we have $\chi_{A(\Sigma_2)}(x) = \frac{n_2}{x-s}$. Also, $x = s = \lambda_k(\Sigma_2)$ is the only pole of $\chi_{A(\Sigma_2)}(x)$, and Theorem 5.5 implies that $\lambda_i(\Sigma_2)$ is an eigenvalue of $\Sigma_1 \odot_{SR} \Sigma_2$ for every i specified in the formulation the theorem. This gives (ii).

The remaining $3n_1$ eigenvalues satisfy $(x-s)(x^2-(\frac{n_2}{x-s}+\lambda_i(\Sigma_1))x-r_1+\lambda_i(\Sigma_1)) = 0$, which after a simple algebraic transformation becomes $x^3 - (s+\lambda_i(\Sigma_1))x^2 - (n_2 - (s+1)\lambda_i(\Sigma_1) + r_1)x + (r_1 - \lambda_i(\Sigma_1))s = 0$, and we are done. □

It is not difficult to see that $\Sigma_1 \odot_{SR} \Sigma_2$ has at most $3l_1 + l_2 + 1$ distinct eigenvalues where l_i is the number of distinct eigenvalues of Σ_i. As in Section 4, we consider the cospectrality.

Remark 5.7. $\Sigma_1' \odot_{SR} \Sigma_2$ are cospectral in the following situations:

(1) Σ_1 and Σ_1' are r-regular and cospectral;

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(2) Σ_1 is regular, and Σ_2 and Σ_2' are cospectral with $\chi_{A(\Sigma_2)}(x) = \chi_{A(\Sigma_2')}(x)$;
(3) If Σ_1 and Σ_1' are cospectral and r-regular, and Σ_2 and Σ_2' are cospectral with $\chi_{A(\Sigma_2)}(x) = \chi_{A(\Sigma_2')}(x)$.

Now we consider the Laplacian characteristic polynomial.

Theorem 5.8. *Let Σ_1 be an r_1-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \dots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and Laplacian eigenvalues $\mu_1(\Sigma_2), \mu_2(\Sigma_2), \dots, \mu_{n_2}(\Sigma_2)$. The Laplacian characteristic polynomial of $\Sigma_1 \odot_{SR} \Sigma_2$ is*

$$\begin{aligned}\psi_{\Sigma_1 \odot_{SR} \Sigma_2}(x) =& x^{m_1} \prod_{i=1}^{n_2} (x - 1 - \mu_i(\Sigma_2))^{n_1} \\ & \cdot \prod_{i=1}^{n_1} \Big(x - r_1 - n_2 - \chi_{L(\Sigma_2)}(x-1) - \Big(1 + \frac{1}{x-2}\Big)(r_1 - \lambda_i(\Sigma_1)) \Big).\end{aligned}$$

Proof. The Laplacian matrix of $\Sigma_1 \odot_{SR} \Sigma_2$ is

$$L(\Sigma_1 \odot_{SR} \Sigma_2) = \begin{pmatrix} L(\Sigma_1) + (r_1 + n_2) I_{n_1} & -B & -I_{n_1} \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & 2I_{m_1} & O_{m_1 \times n_1} \otimes J_{n_2}^{\intercal} \\ -I_{n_1} \otimes J_{n_2} & O_{n_1 \times m_1} \otimes J_{n_2} & I_{n_1} \otimes (L(\Sigma_2) + I_{n_2}) \end{pmatrix},$$

which leads to

$$\begin{aligned}\psi_{\Sigma_1 \odot_{SR} \Sigma_2}(x) &= \det \begin{pmatrix} (x - r_1 - n_2) I_{n_1} - L(\Sigma_1) & B & I_{n_1} \otimes J_{n_2}^{\intercal} \\ B^{\intercal} & (x-2) I_{m_1} & O_{m_1 \times n_1} \otimes J_{n_2}^{\intercal} \\ I_{n_1} \otimes J_{n_2} & O_{n_1 \times m_1} \otimes J_{n_2} & I_{n_1} \otimes ((x-1) I_{n_2} - L(\Sigma_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (x - 1 - \mu_i(\Sigma_2))^{n_1} \det(M),\end{aligned}$$

where

$$\begin{aligned}\det(M) &= \det \begin{pmatrix} (x - r_1 - n_2) I_{n_1} - L(\Sigma_1) - \chi_{L(\Sigma_2)}(x-1) I_{n_1} & B \\ B^{\intercal} & (x-2) I_{m_1} \end{pmatrix} \\ &= \det((x-2) I_{m_1}) \det \left((x - r_1 - n_2) I_{n_1} - L(\Sigma_1) - \chi_{L(\Sigma_2)}(x-1) I_{n_1} - \tfrac{1}{x-2} L(\Sigma_1) \right) \\ &= \det((x-2) I_{m_1}) \det \left((x - r_1 - n_2 - \chi_{L(\Sigma_2)}(x-1)) I_{n_1} - (1 + \tfrac{1}{x-2}) L(\Sigma_1) \right) \\ &= (x-2)^{m_1} \prod_{i=1}^{n_1} \Big(x - r_1 - n_2 - \chi_{L(\Sigma_2)}(x-1) - (1 + \frac{1}{x-2})(r_1 - \lambda_i(\Sigma_1)) \Big),\end{aligned}$$

and the desired result follows. □

We can say more if Σ_2 is co-regular. We recall from Section 2 that if its vertex degree is r_2 and its net degree is s, then we simply say that Σ_2 belongs to the (r_2, s) co-regularity class. In this case $r_2 - s$ is its Laplacian eigenvalue.

Theorem 5.9. *Suppose that, under the assumptions of Theorem 5.8, Σ_2 is co-regular of (r_2, s), with $\mu_k(\Sigma_2) = r_2 - s$ for some fixed k $(1 \leq k \leq n_2)$. The Laplacian spectrum of $\Sigma_1 \odot_{SR} \Sigma_2$ consists of*

(i) 2 *with multiplicity* $m_1 - n_1$,
(ii) $\mu_i(\Sigma_2)$ *with multiplicity* n_1, *for* $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,

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(iii) *and the roots of* $x^3-(2r_1+r_2-s+n_2+3-\lambda_i(\Sigma_1))x^2+(5r_1+2n_2+2-2\lambda_i(\Sigma_1)+(r_2-s)(2r_1+n_2+2-\lambda_i(\Sigma_1))x-(3r_1+2n_2-\lambda_i(\Sigma_1))(1+r_2-s)+2n_2$, *for* $1\le i\le n_1$.

Proof. From the co-regularity of Σ_2, we have $\chi_{L(\Sigma_2)}(x-1)=\frac{n_2}{x-1-r_2+s}$. This leads to

$$\begin{aligned}
&x-r_1-n_2-\chi_{L(\Sigma_2)}(x-1)-\left(1+\frac{1}{x-2}\right)(r_1-\lambda_i(\Sigma_1))\\
&=x-r_1-n_2-\frac{n_2}{x-1-r_2+s}-\left(1+\frac{1}{x-2}\right)(r_1-\lambda_i(\Sigma_1))\\
&=\frac{1}{(x-1-r_2+s)(x-2)}\Big((x-2)(x-(r_1+n_2))(x-(1+r_2-s))-n_2x+2n_2-(x-1)\\
&\quad\cdot(x-(1+r_2-s))(r_1-\lambda_i(\Sigma_1))\Big)\\
&=\frac{1}{(x-1-r_2+s)(x-2)}\Big((x-2)\big(x^2-(r_1+n_2+1+r_2-s)x+(r_1+n_2)(1+r_2-s)\big)\\
&\quad-n_2x+2n_2-\big(x^2-(2+r_2-s)x+1+r_2-s\big)(r_1-\lambda_i(\Sigma_1))\Big)\\
&=\frac{1}{(x-1-r_2+s)(x-2)}\Big(x^3-(r_1+n_2+r_2+1-s+2+r_1-\lambda_i(\Sigma_1))x^2\\
&\quad+((r_1+n_2)(1+r_2-s)+2(r_1+n_2+r_2-s+1)-n_2+(2+r_2-s)(r_1-\lambda_i(\Sigma_1)))x\\
&\quad-2(r_1+n_2)(1+r_2-s)+2n_2-(1+r_2-s)(r_1-\lambda_i(\Sigma_1))\Big)\\
&=\frac{1}{(x-1-r_2+s)(x-2)}\Big(x^3-(2r_1+r_2-s+n_2+3-\lambda_i(\Sigma_1))x^2+(5r_1+2n_2+2-2\lambda_i(\Sigma_1)\\
&\quad+(r_2-s)(2r_1+n_2+2-\lambda_i(\Sigma_1)))x+2n_2-(1+r_2-s)(3r_1+2n_2-\lambda_i(\Sigma_1))\Big).
\end{aligned}$$

Now, the result follows in view of Theorem 5.8. ☐

Remark 5.10. Laplacian cospectral signed graphs are obtained either by taking cospectral r-regular signed graphs Σ_1 and Σ_2, or Laplacian cospectral signed graphs Σ_2 and Σ_2' with $\chi_{L(\Sigma_2)}(x-1)=\chi_{L(\Sigma_2')}(x-1)$, or by combining the previous two settings.

5.2. ***SR*-edge corona.** In this and the following two subsections, we follow the concept of the previous one. In particular, we omit discussions dealing with constructions of cospectral or Laplacian cospectral signed graphs based on the considered products. All of them are analogous to those reported in the previous subsection (see also Section 4). We first treat with the characteristic polynomial.

Theorem 5.11. *Let* Σ_1 *be an* r_1*-regular signed graph with* n_1 *vertices,* m_1 *edges and eigenvalues* $\lambda_1(\Sigma_1),\lambda_2(\Sigma_1),\dots,\lambda_{n_1}(\Sigma_1)$*. Let* Σ_2 *be a signed graph with* n_2 *vertices and eigenvalues* $\lambda_1(\Sigma_2),\lambda_2(\Sigma_2),\dots,\lambda_{n_2}(\Sigma_2)$*. The characteristic polynomial of* $\Sigma_1\ominus_{SR}\Sigma_2$ *is*

$$\begin{aligned}
\phi_{\Sigma_1\ominus_{SR}\Sigma_2}(x)=&(x-\chi_{A(\Sigma_2)}(x))^{m_1-n_1}\prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{m_1}\\
&\cdot\prod_{i=1}^{n_1}\Big(x^2-(\chi_{A(\Sigma_2)}(x)+\lambda_i(\Sigma_1))x+(\chi_{A(\Sigma_2)}(x)+1)\lambda_i(\Sigma_1)-r_1\Big).
\end{aligned}$$

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Proof. From

$$A(\Sigma_1 \ominus_{SR} \Sigma_2) = \begin{pmatrix} A(\Sigma_1) & B & O_{n_1\times m_1}\otimes J_{n_2}^\intercal \\ B^\intercal & O_{m_1\times m_1} & I_{m_1}\otimes J_{n_2}^\intercal \\ O_{m_1\times n_1}\otimes J_{n_2} & I_{m_1}\otimes J_{n_2} & I_{m_1}\otimes A(\Sigma_2) \end{pmatrix},$$

we obtain

$$\begin{aligned} \phi_{\Sigma_1\ominus_{SR}\Sigma_2}(x) &= \det\begin{pmatrix} xI_{n_1}-A(\Sigma_1) & -B & O \\ -B^\intercal & xI_{m_1}-\chi_{A(\Sigma_2)}(x)I_{m_1} & O \\ O_{m_1\times n_1}\otimes J_{n_2} & I_{m_1}\otimes J_{n_2} & I_{m_1}\otimes(xI_{n_2}-A(\Sigma_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{m_1}\det(M), \end{aligned}$$

along with

$$\begin{aligned} \det(M) &= \det\begin{pmatrix} xI_{n_1}-A_{\Sigma_1} & -B \\ -B^\intercal & xI_{m_1}-\chi_{A(\Sigma_2)}(x)I_{m_1} \end{pmatrix} \\ &= (x-\chi_{A(\Sigma_2)}(x))^{m_1-n_1} \\ &\quad\cdot\prod_{i=1}^{n_1}\left(x^2-(\chi_{A(\Sigma_2)}(x)+\lambda_i(\Sigma_1))x+(\chi_{A(\Sigma_2)}(x)+1)\lambda_i(\Sigma_1)-r_1\right), \end{aligned}$$

where we have omitted details and refer the reader to the similar computation in the proofs of Theorems 4.3 and 5.5. The result follows by combining the previous equalities. □

Imposing the net-regularity to Σ_2 we arrive at the following result.

Theorem 5.12. *Suppose that, under the assumptions of Theorem 5.11, Σ_2 is s-net-regular. The spectrum of $\Sigma_1 \ominus_{SR} \Sigma_2$ consists of*

(i) $\lambda_i(\Sigma_2)$ *with multiplicity* $m_1 = \frac{n_1r_1}{2}$, *for* $i \in \{1, 2, \ldots, k-1, k+1, \ldots, n_2\}$,
(ii) *the roots of* $x^2 - sx - n_2$, *both with multiplicity* $m_1 - n_1$,
(ii) *and the roots of* $x^3 - (s + \lambda_i(\Sigma_1))x^2 - (n_2 - (s + 1)\lambda_i(\Sigma_1) + r_1)x + (n_2 - s)\lambda_i(\Sigma_1) + sr_1$, *for* $1 \le i \le n_1$.

Proof. Item (i) follows from Theorem 5.11 as $\chi_{A(\Sigma_2)}(x) = \frac{n_2}{x-s}$ and $x = s = \lambda_k(\Sigma_2)$ is the only pole of $\chi_{A(\Sigma_2)}(x)$.

The remaining $n_1 + 2m_1$ eigenvalues are the roots of $(x - s)(x - \frac{n_2}{x-s})$ and $(x-s)(x^2-(\frac{n_2}{x-s}+\lambda_i(\Sigma_1))x+(\frac{n_2}{x-s}+1)\lambda_i(\Sigma_1)-r_1)$, which leads to items (ii) and (iii). □

Now, the Laplacian characteristic polynomial.

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Theorem 5.13. *Let Σ_1 be an r_1-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \dots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and Laplacian eigenvalues $\mu_1(\Sigma_2), \mu_2(\Sigma_2), \dots, \mu_{n_2}(\Sigma_2)$. The Laplacian characteristic polynomial of $\Sigma_1 \ominus_{SR} \Sigma_2$ is*

$$\psi_{\Sigma_1 \ominus_{SR} \Sigma_2}(x) = (x-2-n_2-\chi_{L(\Sigma_2)}(x-1))^{m_1-n_1} \prod_{i=1}^{n_2} (x-1-\mu_i(\Sigma_2))^{m_1}$$

$$\cdot \prod_{i=1}^{n_1} \Big((x-r_1)(x-2-n_2-\chi_{L(\Sigma_2)}(x-1)) - (x-2-n_2-\chi_{L(\Sigma_2)}(x-1)+1)(r_1-\lambda_i(\Sigma_1)) \Big).$$

Proof. From

$$L(\Sigma_1 \ominus_{SR} \Sigma_2) = \begin{pmatrix} L(\Sigma_1)+r_1 I_{n_1} & -B & O_{n_1\times m_1} \otimes J_{n_2}^{\intercal} \\ -B^{\intercal} & (n_2+2) I_{m_1 \times m_1} & -I_{m_1} \otimes J_{n_2}^{\intercal} \\ O_{m_1 \times n_1} \otimes J_{n_2} & -I_{m_1} \otimes J_{n_2} & I_{m_1} \otimes (L(\Sigma_2)+I_{n_2}) \end{pmatrix},$$

we compute

$$\psi_{\Sigma_1 \ominus_{SR} \Sigma_2}(x) = \prod_{i=1}^{n_2} (x-1-\mu_i(\Sigma_2))^{m_1} \det(M),$$

where

$$\begin{aligned} \det(M) &= \det \begin{pmatrix} (x-r_1)I_{n_1} - L_{\Sigma_1} & B \\ B^{\intercal} & (x-2-n_2)I_{m_1} - \chi_{L(\Sigma_2)}(x-1) I_{m_1} \end{pmatrix} \\ &= \det((x-2-n_2)I_{m_1} - \chi_{L(\Sigma_2)}(x-1)I_{m_1}) \\ &\quad \cdot \det\left((x-r_1)I_{n_1} - L(\Sigma_1) - B((x-2-n_2)I_{m_1} - \chi_{L(\Sigma_2)}(x-1)I_{m_1})^{-1} B^{\intercal}\right) \\ &= \det((x-2-n_2)I_{m_1} - \chi_{L(\Sigma_2)}(x-1)I_{m_1}) \\ &\quad \cdot \det\Big((x-r_1)I_{n_1} - (1+\tfrac{1}{x-2-n_2-\chi_{L(\Sigma_2)}(x-1)}) L(\Sigma_1)\Big) \qquad (5.3) \\ &= (x-2-n_2-\chi_{L(\Sigma_2)}(x-1))^{m_1-n_1} \\ &\quad \cdot \prod_{i=1}^{n_1} \Big((x-r_1)(x-2-n_2-\chi_{L(\Sigma_2)}(x-1)) \\ &\quad -(x-1-n_2-\chi_{L(\Sigma_2)}(x-1))(r_1-\lambda_i(\Sigma_1)) \Big), \qquad (5.4) \end{aligned}$$

and the result follows. ☐

Now, we include the co-regularity.

Theorem 5.14. *Suppose that, under the assumptions of Theorem 5.13, Σ_2 is co-regular signed graph of (r_2, s), with $\mu_k(\Sigma_2) = r_2 - s$ for some fixed k $(1 \le k \le n_2)$. The Laplacian spectrum of $\Sigma_1 \ominus_{SR} \Sigma_2$ consists of*

(i) *$1+\mu_i(\Sigma_2)$ with multiplicity m_1, for $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,*
(ii) *the roots of $x^2-(3+r_2-s+n_2)x+(r_2-s)(2+n_2)+2$, both with multiplicity m_1-n_1,*

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(iii) *and the roots of* $x^3-(3+2r_1+r_2-s+n_2-\lambda_i(\Sigma_1))x^2+(\lambda_i(\Sigma_1)-n_2-r_1+(2+n_2)(2r_1-\lambda_i(\Sigma_1))+(1+r_2-s)(2r_1+n_2+2-\lambda_i(\Sigma_1)))x+n_2(2r_1-\lambda_i(\Sigma_1))-(1+r_2-s)(r_1(n_2+2)+(r_1-\lambda_i(\Sigma_1))(n_2+1))$, *for* $1\le i\le n_1$.

Proof. Since Σ_2 is co-regular, we have $\chi_{L(\Sigma_2)}(x-1)=\frac{n_2}{x-1-r_2+s}$, which yields

$$\begin{aligned} x-2-n_2-\chi_{L(\Sigma_2)}(x-1) &= x-2-n_2-\frac{n_2}{x-1-r_2+s}\\ &= \frac{(x-2-n_2)(x-1-r_2+s)-n_2}{x-1-r_2+s}\\ &= \frac{x^2-(3+r_2-s+n_2)x+(n_2+2)(1+r_2-s)-n_2}{x-1-r_2+s}\\ &= \frac{x^2-(3+r_2-s+n_2)x+(n_2+2)(r_2-s)+2}{x-1-r_2+s}. \end{aligned}$$

We also have

$$\begin{aligned} &(x-r_1)\left(x-2-n_2-\chi_{L(\Sigma_2)}(x-1)\right)-\left(x-1-n_2-\chi_{L(\Sigma_2)}(x-1)\right)(r_1-\lambda_1(\Sigma_1))\\ &=(x-r_1)\left(x-2-n_2-\frac{n_2}{x-1-r_2+s}\right)-\left(x-1-n_2-\frac{n_2}{x-1-r_2+s}\right)(r_1-\lambda_i(\Sigma_1)). \end{aligned}$$

A direct algebraic calculus transforms the previous expression to

$$\begin{aligned} &\frac{1}{x-1-r_2+s}\Big(x^3-(3+r_2-s+n_2+2r_1-\lambda_i(\Sigma_1))x^2+(\lambda_i(\Sigma_1)\\ &-n_2-r_1+(2+n_2)(2r_1-\lambda_i(\Sigma_1))+(1+r_2-s)(2r_1+n_2+2-\lambda_i(\Sigma_1)))x\\ &+n_2(2r_1-\lambda_i(\Sigma_1))-(1+r_2-s)(r_1(n_2+2)+(r_1-\lambda_i(\Sigma_1))(n_2+1))\Big). \end{aligned}$$

The result follows by taking into account Theorem 5.13. □

5.3. SR-vertex neighbourhood corona.

Theorem 5.15. *Let* Σ_1 *be a signed graph with* n_1 *vertices,* m_1 *edges and Laplacian eigenvalues* $\mu_1(\Sigma_1),\mu_2(\Sigma_1),\dots,\mu_{n_1}(\Sigma_1)$. *Let* Σ_2 *be a signed graph with* n_2 *vertices and eigenvalues* $\lambda_1(\Sigma_2),\lambda_2(\Sigma_2),\dots,\lambda_{n_2}(\Sigma_2)$. *The characteristic polynomial of* $\Sigma_1\boxdot_{SR}\Sigma_2$ *is*

$$\begin{aligned} \phi_{\Sigma_1\boxdot_{SR}\Sigma_2}(x)=&x^{m_1-n_1}\prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{n_1}\prod_{i=1}^{n_1}\left(x-\chi_{A(\Sigma_2)}(x)\mu_i(\Sigma_1)\right)\\ &\cdot\det\begin{pmatrix} xI_{n_1}-A(\Sigma_1)-\chi_{A(\Sigma_2)}(x)A^2(\Sigma_1)-(B+\chi_{A(\Sigma_2)}(x)A(\Sigma_1)B)\\ \cdot(xI_{m_1}-\chi_{A(\Sigma_2)}(x)B^\intercal B)^{-1}(B^\intercal+\chi_{A(\Sigma_2)}(x)B^\intercal A(\Sigma_1))\end{pmatrix}, \end{aligned}$$

where B *is the incidence matrix of* Σ_1.

Proof. From

$$A(\Sigma_1\boxdot_{SR}\Sigma_2)=\begin{pmatrix} A(\Sigma_1) & B & A(\Sigma_1)\otimes J_{n_2}^\intercal\\ B^\intercal & O_{m_1\times m_1} & B^\intercal\otimes J_{n_2}^\intercal\\ A(\Sigma_1)\otimes J_{n_2} & B\otimes J_{n_2} & I_{n_1}\otimes A(\Sigma_2)\end{pmatrix}$$

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we obtain

$$\phi_{\Sigma_1 \boxdot_{SR} \Sigma_2}(x) = \prod_{i=1}^{n_2}(x-\lambda_i(\Sigma_2))^{n_1} \cdot \det(M),$$

where

$$\begin{aligned}\det(M) &= \det\begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)A^2(\Sigma_1) & -B - \chi_{A(\Sigma_2)}(x)A(\Sigma_1)B \\ -B^\intercal - \chi_{A(\Sigma_2)}(x)B^\intercal A(\Sigma_1) & xI_{m_1} - \chi_{A(\Sigma_2)}(x)B^\intercal B\end{pmatrix} \\ &= x^{m_1-n_1}\prod_{i=1}^{n_1}\Big(x - \chi_{A(\Sigma_2)}(x)\mu_i(\Sigma_1)\Big) \\ &\quad \cdot \det\begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x)A^2(\Sigma_1) - (B + \chi_{A(\Sigma_2)}(x)A(\Sigma_1)B) \\ \cdot (xI_{m_1} - \chi_{A(\Sigma_2)}(x)B^\intercal B)^{-1}(B^\intercal + \chi_{A(\Sigma_2)}(x)B^\intercal A(\Sigma_1))\end{pmatrix},\end{aligned}$$

and the result follows. □

The Laplacian characteristic polynomial is computed in a similar way. To avoid repetitive proofs, we give less details.

Theorem 5.16. *Let Σ_1 be a r_1-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \ldots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and Laplacian eigenvalues $\mu_1(\Sigma_2), \mu_2(\Sigma_2), \ldots, \mu_{n_2}(\Sigma_2)$. The Laplacian characteristic polynomial of $\Sigma_1 \boxdot_{SR} \Sigma_2$ is*

$$\begin{aligned}\psi_{\Sigma_1 \boxdot_{SR} \Sigma_2}(x) = (x - 2n_2 - 2)^{m_1-n_1}\prod_{i=1}^{n_2}(x - 2r_1 - \mu_i(\Sigma_2))^{n_1} \\ \cdot \prod_{i=1}^{n_1}\Big(x - 2n_2 - 2 - \chi_{L(\Sigma_2)}(x-2r_1)(r_1 - \lambda_i(\Sigma_1)\Big) \\ \cdot \det\begin{pmatrix}(x - r_1(1+n_2))I_{n_1} - L(\Sigma_1) - \chi_{L(\Sigma_2)}(x-2r_1)A^2(\Sigma_1) - (B - \chi_{L(\Sigma_2)}(x-2r_1)A(\Sigma_1)B) \\ \cdot((x-2n_2-2)I_{m_1} - \chi_{L(\Sigma_2)}(x-2r_1)B^\intercal B)^{-1}(B^\intercal - \chi_{L(\Sigma_2)}(x-2r_1)B^\intercal A(\Sigma_1))\end{pmatrix},\end{aligned}$$

where B is the incidence matrix of Σ_1.

Proof. From

$$L(\Sigma_1 \boxdot_{SR} \Sigma_2) = \begin{pmatrix} L(\Sigma_1) + r_1(1+n_2)I_{n_1} & -B & -A(\Sigma_1)\otimes J_{n_2}^\intercal \\ -B^\intercal & (2+2n_2)I_{m_1\times m_1} & -B^\intercal \otimes J_{n_2}^\intercal \\ -A(\Sigma_1)\otimes J_{n_2} & -B\otimes J_{n_2} & I_{n_1}\otimes(L(\Sigma_2) + 2r_1I_{n_2})\end{pmatrix}.$$

we compute

$$\psi_{\Sigma_1 \boxdot_{SR} \Sigma_2}(x) = \prod_{i=1}^{n_2}(x - 2r_1 - \mu_i(\Sigma_2))^{n_1}\det(M),$$

where the latter determinant is

$$\det\begin{pmatrix}(x - r_1(1+n_2))I_{n_1} - L(\Sigma_1) - \chi_{L(\Sigma_2)}(x-2r_1)A^2(\Sigma_1) & B - \chi_{L(\Sigma_2)}(x-2r_1)A(\Sigma_1)B \\ B^\intercal - \chi_{L(\Sigma_2)}(x-2r_1)B^\intercal A(\Sigma_1) & (x-2n_2-2)I_{m_1} - \chi_{L(\Sigma_2)}(x-2r_1)B^\intercal B\end{pmatrix}$$

and it gives the remaining factor of the polynomial. □

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5.4. *SR*-edge neighbourhood corona.

Theorem 5.17. *Let Σ_1 be an r_1-regular signed graph with n_1 vertices, m_1 edges and eigenvalues $\lambda_1(\Sigma_1), \lambda_2(\Sigma_1), \dots, \lambda_{n_1}(\Sigma_1)$. Let Σ_2 be a signed graph with n_2 vertices and eigenvalues $\lambda_1(\Sigma_2), \lambda_2(\Sigma_2), \dots, \lambda_{n_2}(\Sigma_2)$. The characteristic polynomial of $\Sigma_1 \boxminus_{SR} \Sigma_2$ is*

$$\begin{aligned}\phi_{\Sigma_1 \boxminus_{SR} \Sigma_2}(x) = x^{m_1-n_1} \prod_{i=1}^{n_2} (x - \lambda_i(\Sigma_2))^{m_1} \\ \cdot \prod_{i=1}^{n_1} \Big((x^2 - (r_1 \chi_{A(\Sigma_2)}(x) - (1 - \chi_{A(\Sigma_2)}(x)) \lambda_i(\Sigma_1)) x - r_1 + \lambda_i(\Sigma_1) \Big).\end{aligned}$$

Proof. From

$$A(\Sigma_1 \boxminus_{SR} \Sigma_2) = \begin{pmatrix} A(\Sigma_1) & B & B \otimes J_{n_2}^\intercal \\ B^\intercal & O_{m_1 \times m_1} & O_{m_1 \times m_1} \otimes J_{n_2}^\intercal \\ B^\intercal \otimes J_{n_2} & O_{m_1 \times m_1} \otimes J_{n_2} & I_{m_1} \otimes A(\Sigma_2) \end{pmatrix}.$$

we compute

$$\begin{aligned}\phi_{\Sigma_1 \boxminus_{SR} \Sigma_2}(x) &= \det \begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x) BB^\intercal & -B & O \\ -B^\intercal & xI_{m_1} & O \\ -B^\intercal \otimes J_{n_2} & O_{m_1 \times m_1} \otimes J_{n_2} & I_{m_1} \otimes (xI_{n_2} - A(\Sigma_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (x - \lambda_i(\Sigma_2))^{m_1} \det(M),\end{aligned}$$

where

$$\begin{aligned}\det(M) &= \det \begin{pmatrix} xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x) BB^\intercal & -B \\ -B^\intercal & xI_{m_1} \end{pmatrix} \\ &= \det(xI_{m_1}) \det \left(xI_{n_1} - A(\Sigma_1) - \chi_{A(\Sigma_2)}(x) BB^\intercal - B(xI_{m_1})^{-1} B^\intercal) \right) \\ &= \det(xI_{m_1}) \det \left(xI_{n_1} - A(\Sigma_1) - (\tfrac{1}{x} + \chi_{A(\Sigma_2)}(x)) BB^\intercal \right) \\ &= \det(xI_{m_1}) \det \left(xI_{n_1} - A(\Sigma_1) - (\tfrac{1}{x} + \chi_{A(\Sigma_2)}(x))(r_1 I_{n_1} - A(\Sigma_1)) \right) \\ &= x^{m_1} \prod_{i=1}^{n_1} \left((x - \lambda_i(\Sigma_1) - \frac{(1 + x\chi_{A(\Sigma_2)}(x))(r_1 - \lambda_i(\Sigma_1))}{x} \right) \\ &= x^{m_1 - n_1} \prod_{i=1}^{n_1} \Big(x^2 - (r_1 \chi_{A(\Sigma_2)}(x) - (1 - \chi_{A(\Sigma_2)}(x)) \lambda_i(\Sigma_1)) x - r_1 + \lambda_i(\Sigma_1) \Big),\end{aligned}$$

and the proof is completed. □

As before, the net-regularity of Σ_2 provides more information.

Theorem 5.18. *Suppose that, under the assumptions of Theorem 5.17, Σ_2 is s-net-regular, with $\lambda_k(\Sigma_2) = s$ for some fixed k $(1 \leq k \leq n_2)$. The spectrum of $\Sigma_1 \boxminus_{SR} \Sigma_2$ consists of*

(i) 0 *with multiplicity* $m_1 - n_1$,
(ii) s *with multiplicity* $m_1 - n_1$,
(iii) $\lambda_i(\Sigma_2)$ *with multiplicity* n_1, *for* $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,

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(iv) *and the roots of* $x^3+(\lambda_i(\Sigma_1)-s)x^2-(r_1(n_2+1)+(n_2+s-1)\lambda_i(\Sigma_1))x+s(r_1-\lambda_i(\Sigma_1))$, *for* $1\leq i\leq n_1$.

Proof. From the assumption on Σ_2, we have $\chi_{A(\Sigma_2)}=\frac{n_2}{x-s}$. Now,

$$\begin{aligned}
x^2 &- (r_1\chi_{A(\Sigma_2)}(x)-(1-\chi_{A(\Sigma_2)}(x))\lambda_i(\Sigma_1))x-r_1+\lambda_i(\Sigma_1)\\
&= x^2+(\lambda_i(\Sigma_1)-\frac{r_1n_2}{x-s}-\frac{n_2\lambda_i(\Sigma_1)}{x-s})x-r_1+\lambda_i(\Sigma_1)\\
&= \frac{1}{x-s}\Big(x^2(x-s)+(\lambda_i(\Sigma_1)(x-s)-r_1n_2-n_2\lambda_i(\Sigma_1))x+(\lambda_i(\Sigma_1)-r_1)(x-s)\Big)\\
&= \frac{1}{x-s}\big(x^3+(\lambda_i(\Sigma_1)-s)x^2+(\lambda_i(\Sigma_1)-n_2\lambda_i(\Sigma_1)-\lambda_i(\Sigma_1)s-r_1n_2-r_1)x+r_1s-s\lambda_i(\Sigma_1)\big)\\
&= \frac{1}{x-s}\Big(x^3+(\lambda_i(\Sigma_1)-s)x^2+(\lambda_i(\Sigma_1)(1-n_2-s)-r_1(n_2+1))x+(r_1-\lambda_i(\Sigma_1))s\Big),
\end{aligned}$$

which, in view of Theorem 5.17, leads to the desired result. □

It remains to compute the Laplacian characteristic polynomial.

Theorem 5.19. *Let* Σ_1 *be an* r*-regular signed graph with* n_1 *vertices,* m_1 *edges and eigenvalues* $\lambda_1(\Sigma_1),\lambda_2(\Sigma_1),\dots,\lambda_{n_1}(\Sigma_1)$. *Let* Σ_2 *be a signed graph with* n_2 *vertices and Laplacian eigenvalues* $\mu_1(\Sigma_2),\mu_2(\Sigma_2),\dots,\mu_{n_2}(\Sigma_2)$. *The Laplacian characteristic polynomial of* $\Sigma_1\boxminus_{SR}\Sigma_2$ *is*

$$\begin{aligned}
\psi_{\Sigma_1\boxminus_{SR}\Sigma_2}(x) &= (x-2)^{m_1}\prod_{i=1}^{n_2}(x-2-\mu_i(\Sigma_2))^{m_1}\\
&\quad\cdot\prod_{i=1}^{n_1}\Big(x-r_1(1+n_2)-(1+\chi_{L(\Sigma_2)}(x-2)+\frac{1}{x-2})(r_1-\lambda_i(\Sigma_1))\Big).
\end{aligned}$$

Proof. From

$$L(\Sigma_1\boxminus_{SR}\Sigma_2)=\begin{pmatrix} L(\Sigma_1)+r_1(1+n_2)I_{n_1} & -B & -B\otimes J_{n_2}^{\intercal}\\ -B^{\intercal} & 2I_{m_1\times m_1} & O_{m_1\times m_1}\otimes J_{n_2}^{\intercal}\\ -B^{\intercal}\otimes J_{n_2} & O_{m_1\times m_1}\otimes J_{n_2} & I_{m_1}\otimes(L(\Sigma_2)+2I_{n_2})\end{pmatrix}$$

we compute

$$\psi_{\Sigma_1\boxminus_{SR}\Sigma_2}(x) = \prod_{i=1}^{n_2}(x-2-\mu_i(\Sigma_2))^{m_1}\det(M),$$

where

$$\begin{aligned}
\det(M) &= \det\begin{pmatrix}(x-r_1(1+n_2))I_{n_1}-L(\Sigma_1)-\chi_{L(\Sigma_2)}(x-2)BB^{\intercal} & B\\ B^{\intercal} & (x-2)I_{m_1}\end{pmatrix}\\
&= \det((x-2)I_{m_1})\det\Big((x-r_1(1+n_2))I_{n_1}-(1+\chi_{L(\Sigma_2)}(x-2))L(\Sigma_1)-\tfrac{L(\Sigma_1)}{x-2})\Big)\\
&= (x-2)^{m_1}\prod_{i=1}^{n_1}\Big(x-r_1(1+n_2)-(1+\chi_{L(\Sigma_2)}(x-2)+\frac{1}{x-2})(r_1-\lambda_i(\Sigma_1))\Big),
\end{aligned}$$

and the result follows. □

Finally, we take into account the co-regularity of Σ_2.

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Theorem 5.20. *Suppose that, under the assumptions of Theorem 5.19, Σ_2 is a co-regular signed graph of (r_2, s), with $\mu_k(\Sigma_2) = r_2 - s$ for some fixed k $(1 \leq k \leq n_2)$. The Laplacian spectrum of $\Sigma_1 \boxminus_{SR} \Sigma_2$ consists of*

(i) 2 *with multiplicity* $m_1 - n_1$,
(ii) $2 + r_2 - s$ *with multiplicity* $m_1 - n_1$,
(iii) $2 + \mu_i(\Sigma_2)$ *with multiplicity* m_1, *for* $i \in \{1, 2, \dots, k-1, k+1, \dots, n_2\}$,
(iv) *and the roots of* $x^3 - (4+2r_1+r_2-s+r_1n_2-\lambda_i(\Sigma_1))x^2 + ((4+r_2-s)(2+2r_1+r_1n_2-\lambda_i(\Sigma_1)) - (n_2+1)(r_1-\lambda_i(\Sigma_1)) - 4)x + (2+r_2-s)(\lambda_i(\Sigma_1) - (3+2n_2)r_1) + 2n_2(r_1-\lambda_i(\Sigma_1))$, *for* $1 \leq i \leq n_1$.

Proof. The assumption on Σ_2 gives $\chi_{L(\Sigma_2)}(x-2) = \frac{n_2}{x-2-r_2+s}$. We compute

$$\begin{aligned}
x &- r_1(1+n_2) - \Big(1 + \chi_{L(\Sigma_2)}(x-2) + \frac{1}{x-2}\Big)(r_1 - \lambda_i(\Sigma_1)) \\
&= x - r_1(1+n_2) - \Big(1 + \frac{n_2}{x-2-r_2+s} + \frac{1}{x-2}\Big)(r_1 - \lambda_i(\Sigma_1)) \\
&= \frac{1}{(x-2)(x-2-r_2+s)}\Big(x(x^2 - (4+r_2-s)x + 2(2+r_2-s)) - r_1(1+n_2)(x^2 - (4+r_2-s)x \\
&\quad +2(2+r_2-s)) + (x^2 - (4+r_2-s)x + 2(2+r_2-s) + n_2x - 2n_2 + x - 2 - r_2 + s)(r_1 - \lambda_i(\Sigma_1))\Big) \\
&= \frac{1}{(x-2)(x-2-r_2+s)}\Big(x^3 - (4+r_2-s+r_1(1+n_2) + (r_1-\lambda_i(\Sigma_1)))x^2 \\
&\quad +(2(2+r_2-s) + r_1(1+n_2)(4+r_2-s) + (4+r_2-s)(r_1-\lambda_i(\Sigma_1) - (r_1-\lambda_i(\Sigma_1))(1+n_2))x \\
&\quad -2r_1(1+n_2)(2+r_2-s) - 2(2+r_2-s)(r_1-\lambda_i(\Sigma_1)) + (2n_2+2+r_2-s)(r_1-\lambda_i(\Sigma_1))\Big) \\
&= \frac{1}{(x-2)(x-2-r_2+s)}\Big(x^3 - (4+2r_1+r_2-s+r_1n_2-\lambda_i(\Sigma_1))x^2 + ((4+r_2-s)(2+2r_1+r_1n_2 \\
&\quad -\lambda_i(\Sigma_1)) - (n_2+1)(r_1-\lambda_i(\Sigma_1)) - 4)x + (2+r_2-s)(\lambda_i(\Sigma_1) - (3+2n_2)r_1) + 2n_2(r_1-\lambda_i(\Sigma_1))\Big),
\end{aligned}$$

and the result follows in view of Theorem 5.19. ☐

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