Operators and Matrices
Volume 18, Number 3 (2024), 697–710 doi:10.7153/oam-2024-18-42

NORM OF COMPOSITION OPERATOR ON MIXED NORM SPACES

DUŠICA DMITROVIĆ

(*Communicated by E. Fricain*)

Abstract. Computation of the exact norm of the composition operator acting on the spaces of holomorphic functions has shown to be difficult to perform. In this paper, we provide estimates of the norm of this operator acting on the mixed norm space $H^{p,q,\alpha}$. For some values of parameters p,q and α, the proposed upper bound of the norm generalizes the upper bound of the norm of the composition operator acting on the weighted Bergman space.

1. Introduction

Throughout the paper, $\mathbb{C}$ denotes the complex plane and $\mathrm{Hol}(\mathbb{D})$ denotes the space of all holomorphic functions on the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. For $0<p\leqslant\infty$ and $0<q,\alpha<\infty$, the mixed norm space $H^{p,q,\alpha}$ is the space of all functions $f\in\mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{p,q,\alpha}=\Big(2\alpha q\int_0^1 r(1-r^2)^{\alpha q-1}M_p^q(r,f)\,dr\Big)^{\frac{1}{q}}<\infty,$$

where for $0<p<\infty$,

$$M_p(r,f)=\Big(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\Big)^{\frac{1}{p}}$$

and $p=\infty$,

$$M_\infty(r,f)=\sup_{\theta\in[0,2\pi)}|f(re^{i\theta})|.$$

As in [8], we define the space $H^{p,\infty,\alpha}$ for $0<p\leqslant\infty$ and $0<\alpha<\infty$ as the set of all functions $f\in\mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{p,\infty,\alpha}=\sup_{0<r<1}(1-r^2)^{\alpha}M_p(r,f)<\infty.$$

Mixed norm spaces were introduced and studied by Flett, [5, 6]. Since then, these spaces have been studied by many authors. They form a family of complete spaces that

Mathematics subject classification (2020): 30H20, 46E30, 47A30.
Keywords and phrases: Composition operator, norm, mixed norm space.
The author would like to thank her advisor Dr Boban Karapetrović for his valuable help and comments during the work on this paper. The author is supported in part by the Serbian Ministry of Science, Technological Development and Innovation, contract number 451-03-47/2023-01/200104.

contain Bergman spaces. Specifically, the space $H^{p,p,\frac{\gamma+1}{p}}$ coincides with the weighted Bergman space A^p_γ for $\gamma > -1$ and $0 < p < \infty$. The norm of a function $f \in A^p_\gamma$ is

$$\|f\|_{A^p_\gamma} = \Big(\frac{\gamma+1}{\pi}\int_{\mathbb{D}}(1-|z|^2)^\gamma |f(z)|^p\,dm(z)\Big)^{\frac{1}{p}},$$

where dm is the Euclidean area measure in the complex plane. Since the integral in the norm is evaluated on $\mathbb{D}$, it is simpler to estimate the norm in this case than in the case of an arbitrary mixed norm space $H^{p,q,\alpha}$ when $p \neq q$. See [4, 7, 13].

A holomorphic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ induces the composition operator C_φ on $\mathrm{Hol}(\mathbb{D})$, defined by $\mathrm{C}_\varphi f = f\circ\varphi$ for $f \in \mathrm{Hol}(\mathbb{D})$. Research on composition operators on spaces of holomorphic functions starts with the work of Nordgren [10]. It is difficult to find the exact norm of composition operator C_φ, but the problem has been solved when φ is a linear fractional map in certain function spaces. See [3, 12]. In the case of the weighted Bergman space, in [13], the norm of C_φ is estimated by

$$\frac{1}{(1-|\varphi(0)|^2)^{\frac{\gamma+2}{p}}} \leqslant \|\mathrm{C}_\varphi\|_{A^p_\gamma\to A^p_\gamma} \leqslant \Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{\frac{\gamma+2}{p}}.$$

This motivates us to estimate the norm of the composition operator C_φ acting on the mixed norm space $H^{p,q,\alpha}$ in a similar form by

$$C_1\frac{1}{(1-|\varphi(0)|^2)^{\alpha+\frac{1}{p}}} \leqslant \|\mathrm{C}_\varphi\|_{H^{p,q,\alpha}\to H^{p,q,\alpha}} \leqslant C_2\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{\alpha+\frac{1}{p}},$$

where these constants C_1 and C_2 depend on p,q,α and $|\varphi(0)|$. If $a=\infty$ we define $\frac{1}{a} := 0$, $\beta a = \infty$ for $0<\beta<\infty$. The main result of this paper is the following:

THEOREM 1. *Let* $0 < p,q \leqslant \infty$, $0<\alpha<\infty$ *and* $\varphi : \mathbb{D}\to\mathbb{D}$ *be holomorphic then*

$$C_1\frac{1}{(1-|\varphi(0)|^2)^{\alpha+\frac{1}{p}}} \leqslant \|\mathrm{C}_\varphi\|_{H^{p,q,\alpha}\to H^{p,q,\alpha}} \leqslant C_2\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{\alpha+\frac{1}{p}},$$

where

$$C_1 = \begin{cases} \dfrac{\Gamma\Big(\frac{\alpha p+\frac{p}{q}+1}{2}\Big)^{\frac{2}{p}}}{\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}}, & 0<p,q<\infty,\ \frac{q}{p}\leqslant 1;\\[2ex] \dfrac{\Gamma\Big(\frac{\alpha p+\frac{p}{q}+1}{2}\Big)^{\frac{2}{p}}}{2^{\alpha+\frac{3}{p}}\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}}, & 0<p,q<\infty,\ \frac{q}{p}>1;\\[2ex] \dfrac{\Gamma\Big(\frac{\alpha p+1}{2}\Big)^{\frac{2}{p}}}{2^{\alpha}\Gamma(\alpha p)^{\frac{1}{p}}}, & 0<p<\infty,\ q=\infty;\\[2ex] \Big(\frac{2\alpha q|\varphi(0)|}{1-|\varphi(0)|^2}+\Big(\frac{3\sqrt{3}}{4}\Big)^{\alpha q}\Big)^{-\frac{1}{q}}, & p=\infty,\ 0<q<\infty;\\ 1, & p=\infty,\ q=\infty, \end{cases}$$

and

$$C_2 = \begin{cases} 1, & \alpha q \geqslant 1, \frac{q}{p} \leqslant 1 \text{ or } \alpha q < 1, \frac{q}{p} \leqslant \frac{1}{2} \text{ or } p = \infty, q = \infty; \\ \left(\frac{3+|\varphi(0)|}{1+|\varphi(0)|}\right)^{\frac{1}{p}-\frac{1}{q}}, & \alpha q \geqslant 1, \frac{q}{p} > 1; \\ \left(\frac{1+|\varphi(0)|}{1+3|\varphi(0)|}\right)^{\alpha-\frac{1}{q}}, & \alpha q < 1, \frac{1}{2} < \frac{q}{p} \leqslant 1; \\ (1+|\varphi(0)|)^{\alpha-\frac{1}{p}}, & \alpha q < 1, \frac{q}{p} > 1. \end{cases}$$

2. Preliminaries

In the case where parameters $0 < p, q < \infty$, we are going to estimate $|\varphi(z)|$ by function R which depends on $|z|$ when $0 < |z| < 1$. It will be useful to separate function R from $|\varphi(0)|$ in certain cases, so we give two estimates for $|\varphi(z)|$, where the second one was initially introduced in [11]. The first estimate,

$$|\varphi(z)| \leqslant \frac{|z| + |\varphi(0)|}{1 + |\varphi(0)||z|} \tag{1}$$

is a consequence of the Schwarz-Pick lemma. The second estimate,

$$|\varphi(z)| \leqslant \frac{(1 - |\varphi(0)|)|z| + 2|\varphi(0)|}{1 + |\varphi(0)|} \tag{2}$$

is easy to check from the first one.

LEMMA 1. *Let* $f \in \mathrm{Hol}(\mathbb{D})$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$ *be holomorphic. Then for* $0 < p < \infty$ *and* $0 < r < 1$

$$M_p^p(r, f \circ \varphi) \leqslant \frac{R + |\varphi(0)|}{R - |\varphi(0)|} M_p^p(R, f),$$

where $|\varphi(z)| \leqslant R < 1$ *for all* $|z| = r$.

Proof. Let $|\varphi(0)| = c$. Since $|f|^p$ is subharmonic on $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ and continuous on $\overline{\mathbb{D}_R}$, there is a continuous function u defined on $\overline{\mathbb{D}_R}$, such that u is harmonic on $\mathbb{D}_R$, $|f(z)|^p \leqslant u(z)$ for $z \in \mathbb{D}_R$ and $|f(z)|^p = u(z)$ for $|z| = R$. Then

$$\begin{aligned} M_p^p(r, f \circ \varphi) &= \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p \, d\theta \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} u(\varphi(re^{i\theta})) \, d\theta = u(\varphi(0)). \end{aligned} \tag{3}$$

By using the Harnack inequality in (3), we obtain

$$\begin{aligned} M_p^p(r, f \circ \varphi) \leqslant& \frac{R+c}{R-c} u(0) = \frac{R+c}{R-c} \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \, d\theta \\ =& \frac{R+c}{R-c} \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|^p \, d\theta. \end{aligned}$$

Hence,

$$M_p^p(r, f\circ\varphi) \leqslant \frac{R+c}{R-c} M_p^p(R, f). \quad \square$$

2.1. Pointwise estimates

We first obtain estimates of the norm of the functional $F_z : H^{p,q,\alpha} \to \mathbb{C}$ defined by $F_z f = f(z)$ for $z \in \mathbb{D}$.

LEMMA 2. *Let $z \in \mathbb{D}$ then*

$$L_1 \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}} \leqslant \|F_z\| \leqslant L_2 \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}},$$

where

$$L_1 = \begin{cases} \dfrac{\Gamma\left(\frac{\alpha p+\frac{p}{q}+1}{2}\right)^{\frac{2}{p}}}{\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}}, & 0<p<\infty, 0<q\leqslant\infty; \\ \left(\frac{2\alpha q|z|}{1-|z|^2} + \left(\frac{3\sqrt{3}}{4}\right)^{\alpha q}\right)^{-\frac{1}{q}}, & p=\infty, 0<q<\infty; \\ 1, & p=\infty, q=\infty, \end{cases}$$

and

$$L_2 = \begin{cases} 1, & 0<p,q<\infty,\ \frac{p}{q}\geqslant 1 \text{ or } p=\infty, 0<q\leqslant\infty; \\ 2^{\alpha+\frac{3}{p}}, & 0<p,q<\infty,\ \frac{p}{q}<1; \\ 2^{\alpha}(1+3|z|)^{\frac{1}{p}}(1+|z|)^{\frac{1}{p}}, & 0<p<\infty, q=\infty. \end{cases}$$

Proof. Since the norm $\|F_z\|$ is equal to $\sup \frac{|F_z f|}{\|f\|_{p,q,\alpha}}$, where the supremum is taken over all nonzero functions $f \in H^{p,q,\alpha}$, and $\|\cdot\|_{p,q,\alpha}$ is rotational invariant, then the norm of F_z is also rotational invariant because

$$\frac{|f(z)|}{\|f\|_{p,q,\alpha}} = \frac{|f\circ\mathscr{R}_{\arg z}(|z|)|}{\|f\|_{p,q,\alpha}}$$

where $\mathscr{R}_\theta z = ze^{i\theta}$ and $f \in H^{p,q,\alpha}$. To find the lower bound of the norm F_z, where $|z| = r_0$, it is enough to estimate from below the expression $\frac{|F_{r_0} f|}{\|f\|_{p,q,\alpha}}$ for some function $f \in H^{p,q,\alpha}$. To find the upper bound of $\|F_z\|$, we will estimate $|f(z)|$ for all $f \in H^{p,q,\alpha}$ when $z \in \mathbb{D}$.

- **Case $0 < p,q < \infty$:**

  Let $z \in \mathbb{D}$ be arbitrary and $|z| = r_0$. Since function $f(w) = \frac{1}{(1-r_0 w)^{\alpha+\frac{1}{p}+\frac{1}{q}}}$ for $w \in \mathbb{D}$ is in $H^{p,q,\alpha}$ and the norm of function f is equal to

$$\|f\|_{p,q,\alpha} = \left(2\alpha q \int_0^1 r(1-r^2)^{\alpha q-1} M_p^q(r,f)\,dr\right)^{\frac{1}{q}},$$

by using Theorem 1.3 from [9] we obtain the following estimate

$$M_p^p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-r_0re^{i\theta}|^{\alpha p+\frac{p}{q}+1}} \leqslant \frac{\Gamma(\alpha p+\frac{p}{q})}{\Gamma(\frac{\alpha p+\frac{p}{q}+1}{2})^2}\frac{1}{(1-r_0^2r^2)^{\alpha p+\frac{p}{q}}}.$$

If $C = \frac{\Gamma(\alpha p+\frac{p}{q})}{\Gamma(\frac{\alpha p+\frac{p}{q}+1}{2})}$ we get

$$\|f\|_{p,q,\alpha}^q \leqslant C^{\frac{q}{p}} 2\alpha q\int_0^1 r\Big(\frac{1-r^2}{1-r_0^2r^2}\Big)^{\alpha q-1}\frac{dr}{(1-r_0^2r^2)^2}.$$

After the change of variable $t = \frac{1-r^2}{1-r_0^2r^2}$, we have

$$\|f\|_{p,q,\alpha}^q \leqslant \frac{C^{\frac{q}{p}}}{1-r_0^2}\alpha q\int_0^1 t^{\alpha q-1}\,dt = \frac{C^{\frac{q}{p}}}{1-r_0^2}.$$

Hence, after replacing $|z| = r_0$, we obtain the lower bound of the norm

$$\|F_z\| \geqslant \frac{|f(r_0)|}{\|f\|_{p,q,\alpha}} \geqslant \frac{\Gamma\Big(\frac{\alpha p+\frac{p}{q}+1}{2}\Big)^{\frac{2}{p}}}{\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}}\frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}.$$

To find the upper bound of $\|F_z\|$, we need to estimate $|f(z)|$ for all $f \in H^{p,q,\alpha}$ for $z \in \mathbb{D}$, to do that, we use the method similar to the one used in [1]. Through the application of the subharmonic property of the function $|f|^p$, for $z = r_0e^{i\theta}$ and $r > r_0$ we have

$$\begin{aligned} |f(r_0e^{i\theta})|^p \leqslant& \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \frac{r^2-r_0^2}{|r-r_0e^{i(\theta-t)}|^2}\,dt \\ \leqslant& \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \frac{r+r_0}{r-r_0}\,dt. \end{aligned} \tag{4}$$

Hence,

$$|f(z)|^q\Big(\frac{r-r_0}{r+r_0}\Big)^{\frac{q}{p}} \leqslant M_p^q(r,f).$$

After integration of the previous inequality with respect to the measure $2\alpha qr(1-r^2)^{\alpha q-1}\,dr$ on the interval $(A,1)$, where $A \in (r_0,1)$, we get

$$|f(z)|^q\int_A^1 2\alpha qr(1-r^2)^{\alpha q-1}\Big(\frac{r-r_0}{r+r_0}\Big)^{\frac{q}{p}}\,dr \leqslant \|f\|_{p,q,\alpha}^q.$$

Since function $\frac{r-r_0}{r+r_0}$ is increasing for $r \in (r_0,1)$ and A is arbitrary in $(r_0,1)$ we can choose $A = \frac{1+r_0}{2}$

$$\begin{aligned} 2\alpha q\int_A^1 r(1-r^2)^{\alpha q-1}\Big(\frac{r-r_0}{r+r_0}\Big)^{\frac{q}{p}}\,dr \geqslant& \Big(\frac{A-r_0}{A+r_0}\Big)^{\frac{q}{p}}(1-A^2)^{\alpha q} \\ =& \Big(\frac{1-r_0}{1+3r_0}\Big)^{\frac{q}{p}}\Big(\frac{1-r_0}{2}\Big)^{\alpha q}\Big(\frac{3+r_0}{2}\Big)^{\alpha q}. \end{aligned}$$

By using $\frac{3+r_0}{2} \geqslant 1+r_0$ and $\frac{1}{1+3r_0} \geqslant \frac{1+r_0}{8}$, we have

$$\left(\frac{1-r_0}{1+3r_0}\right)^{\frac{q}{p}}\left(\frac{1-r_0}{2}\right)^{\alpha q}\left(\frac{3+r_0}{2}\right)^{\alpha q} \geqslant (1-r_0^2)^{\alpha q+\frac{q}{p}}\frac{1}{2^{\alpha q+3\frac{q}{p}}}.$$

Hence, first estimate of the upper bound of the norm in case when $0<p,q<\infty$ is

$$\|F_z\| \leqslant 2^{\alpha+\frac{3}{p}}\frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}.$$

From Proposition 1 in [8], the norm $\|\cdot\|_{p,q,\alpha}$ is decreasing with respect to parameter q and from the main theorem from [8], $H^{p,q,\alpha} \subseteq H^{p,p,\alpha}$. Then in case when $p \geqslant q$ we get

$$\|f\|_{p,q,\alpha} \geqslant \|f\|_{p,p,\alpha}.$$

The space $H^{p,p,\alpha}$ is actually the Bergman weighted space $A^p_{\alpha p-1}$. It is well known that for $f \in A^p_\gamma$ and $z \in \mathbb{D}$ holds

$$|f(z)| \leqslant \frac{\|f\|_{A^p_\gamma}}{(1-|z|^2)^{\frac{\gamma+2}{p}}}.$$

Hence, for $f \in H^{p,q,\alpha}$

$$|f(z)| \leqslant \frac{\|f\|_{p,q,\alpha}}{(1-|z|^2)^{\alpha+\frac{1}{p}}},$$

or $\|F_z\| \leqslant \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}$ in case $0<p,q<\infty$ and $p \geqslant q$.

- **Case $0<p<\infty$, $q=\infty$:**

Let $z \in \mathbb{D}$ be arbitrary such that $|z|=r_0$. For $f(w)=\frac{1}{(1-r_0w)^{\alpha+\frac{1}{p}}}$, $w \in \mathbb{D}$, from Theorem 1.3 in [9] we have

$$M_p^p(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-r_0re^{i\theta}|^{\alpha p+1}} \leqslant \frac{\Gamma(\alpha p)}{\Gamma(\frac{\alpha p+1}{2})^2}\frac{1}{(1-r_0^2r^2)^{\alpha p}}.$$

Then

$$\|f\|_{p,\infty,\alpha} \leqslant \left(\frac{\Gamma(\alpha p)}{\Gamma(\frac{\alpha p+1}{2})^2}\right)^{\frac{1}{p}}\sup_{0<r<1}\left(\frac{1-r^2}{1-r_0^2r^2}\right)^{\alpha}=\left(\frac{\Gamma(\alpha p)}{\Gamma(\frac{\alpha p+1}{2})^2}\right)^{\frac{1}{p}}.$$

Hence,

$$\|F_z\| \geqslant \frac{|f(r_0)|}{\|f\|_{p,\infty,\alpha}} \geqslant \left(\frac{\Gamma(\frac{\alpha p+1}{2})^2}{\Gamma(\alpha p)}\right)^{\frac{1}{p}}\frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}.$$

To estimate the norm from above, we use inequality (4) in an arbitrary point $z=r_0e^{i\theta} \in \mathbb{D}$. Hence, for arbitrary $f \in H^{p,\infty,\alpha}$ and $r>r_0$

$$\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{p}}|f(z)| \leqslant M_p(r,f).$$

After multiplication of the previous inequality with $(1-r^2)^{\alpha}$ and taking the supremum over $r_0 < r < 1$ we get

$$|f(z)| \sup_{r_0<r<1} (1-r^2)^{\alpha}\Big(\frac{r-r_0}{r+r_0}\Big)^{\frac{1}{p}} \leqslant \sup_{r_0<r<1} M_p(r,f)(1-r^2)^{\alpha} \leqslant \|f\|_{p,\infty,\alpha}.$$

Assigning $r = \frac{1+r_0}{2}$ we conclude

$$\sup_{r_0<r<1} (1-r^2)^{\alpha}\Big(\frac{r-r_0}{r+r_0}\Big)^{\frac{1}{p}} \geqslant \frac{(1-r_0^2)^{\alpha+\frac{1}{p}}}{2^{\alpha}(1+3r_0)^{\frac{1}{p}}(1+r_0)^{\frac{1}{p}}}.$$

Hence, we obtain

$$\|F_z\| \leqslant 2^{\alpha}(1+3|z|)^{\frac{1}{p}}(1+|z|)^{\frac{1}{p}} \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}} \leqslant 2^{\alpha+\frac{3}{p}} \frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}.$$

- **Case $p=\infty, 0<q<\infty$:**

For $z \in \mathbb{D}$ such that $|z| = r_0$, the function $f(w) = \frac{1}{(1-r_0 w)^{2\alpha}}$ for $w \in \mathbb{D}$ is in $H^{\infty,q,\alpha}$ and $M_{\infty}(r,f) = \frac{1}{(1-r_0 r)^{2\alpha}}$. Let for $0<r<1$

$$g(r) = r(1-r^2)^{\alpha q-1} M_{\infty}^{q}(r,f) = r\Big(\frac{1-r^2}{(1-r_0 r)^2}\Big)^{\alpha q-1} \frac{1}{(1-r_0 r)^2},$$

then

$$\|f\|_{\infty,q,\alpha}^{q} = 2\alpha q\Big(\underbrace{\int_0^{r_0} g(r)\,dr}_{I_1} + \underbrace{\int_{r_0}^{\frac{2r_0}{1+r_0^2}} g(r)\,dr}_{I_2} + \underbrace{\int_{\frac{2r_0}{1+r_0^2}}^{1} g(r)\,dr}_{I_3}\Big).$$

In I_1 and I_2, we aim to apply a substitution by introducing the variable $u = \frac{1-r^2}{(1-r_0 r)^2}$. When $0<r<r_0$, $r = \frac{ur_0-\sqrt{ur_0^2-u+1}}{ur_0^2+1}$ for $1<u<\frac{1}{1-r_0^2}$ and when $r_0 < r < \frac{2r_0}{1+r_0^2}$, $r = \frac{ur_0+\sqrt{ur_0^2-u+1}}{ur_0^2+1}$ for $1<u<\frac{1}{1-r_0^2}$. After applying the change of variable in both I_1 and I_2 and using the monotonicity of $u^{\alpha q}$, we obtain

$$\begin{aligned} I_1+I_2 &= \int_1^{\frac{1}{1-r_0^2}} u^{\alpha q-1} \frac{r_0 u}{\sqrt{ur_0^2-u+1}(1+ur_0^2)}\,du \leqslant \frac{1}{2}\int_1^{\frac{1}{1-r_0^2}} u^{\alpha q} \frac{du}{\sqrt{ur_0^2-u+1}} \\ &\leqslant \frac{1}{2(1-r_0^2)^{\alpha q}} \int_1^{\frac{1}{1-r_0^2}} \frac{du}{\sqrt{ur_0^2-u+1}} = \frac{r_0}{(1-r_0^2)^{\alpha q+1}}. \end{aligned}$$

Since $\frac{1}{1-r_0 r}$ is an increasing function and $\alpha q > 0$

$$I_3 = \int_{\frac{2r_0}{1+r_0^2}}^{1} r\frac{(1-r^2)^{\alpha q-1}}{(1-r_0 r)^{2\alpha q}}\,dr \leqslant \frac{1}{(1-r_0)^{2\alpha q}}\int_{\frac{2r_0}{1+r_0^2}}^{1} r(1-r^2)^{\alpha q-1}\,dr$$
$$=\frac{1}{(1-r_0)^{2\alpha q}}\frac{1}{2\alpha q}\frac{(1-r_0^2)^{2\alpha q}}{(1+r_0^2)^{2\alpha q}} = \frac{1}{2\alpha q}\frac{(1+r_0)^{2\alpha q}}{(1+r_0^2)^{2\alpha q}}$$
$$<\frac{1}{2\alpha q}\Big(\frac{3\sqrt{3}}{4}\Big)^{\alpha q}\frac{1}{(1-r_0^2)^{\alpha q}}.$$

Hence,

$$\|f\|_{\infty,q,\alpha}^q \leqslant \frac{1}{(1-r_0^2)^{\alpha q}}\Big(\frac{2\alpha q r_0}{1-r_0^2}+\Big(\frac{3\sqrt{3}}{4}\Big)^{\alpha q}\Big),$$

so we obtain the lower bound of the norm

$$\|F_z\|_{\infty,q,\alpha} \geqslant \frac{f(r_0)}{\|f\|_{p,q,\alpha}} \geqslant \frac{1}{(1-|z|^2)^{\alpha}}\Big(\frac{2\alpha q|z|}{1-|z|^2}+\Big(\frac{3\sqrt{3}}{4}\Big)^{\alpha q}\Big)^{-\frac{1}{q}}.$$

For the upper bound, for arbitrary $f \in H^{\infty,q,\alpha}$ and $z \in \mathbb{D}$ from the maximum principle, we have

$$|f(z)|^q \leqslant \sup_{\theta\in[0,2\pi)} |f(re^{i\theta})|^q \tag{5}$$

for all $r \geqslant |z|$. By integrating inequality (5) with respect to the measure $2\alpha q r(1-r^2)^{\alpha q-1}$ on the interval $[\,|z|,1]$, we get

$$(1-|z|^2)^{\alpha q}|f(z)|^q \leqslant 2\alpha q\int_{|z|}^{1} r(1-r^2)^{\alpha q-1}M_\infty^q(r,f)\,dr \leqslant \|f\|_{\infty,q,\alpha}^q.$$

Hence,

$$\|F_z\| \leqslant \frac{1}{(1-|z|^2)^{\alpha}}.$$

- **Case $p=\infty, q=\infty$:**

For $f(w) = \frac{1}{(1-w^2)^{\alpha}}$ when $w \in \mathbb{D}$, we have

$$\|f\|_{\infty,\infty,\alpha} = \sup_{0<r<1}(1-r^2)^{\alpha}\sup_{\theta\in[0,2\pi)}\frac{1}{|1-r^2e^{2i\theta}|^{\alpha}} = \sup_{0<r<1}\Big(\frac{1-r^2}{1-r^2}\Big)^{\alpha} = 1.$$

Then, for arbitrary $z \in \mathbb{D}$ such that $|z| = r_0$, we obtain

$$\|F_z\| \geqslant \frac{|f(r_0)|}{\|f\|_{\infty,\infty,\alpha}} = \frac{1}{(1-|z|^2)^{\alpha}}.$$

For an arbitrary function $f \in H^{\infty,\infty,\alpha}$ and $z \in \mathbb{D}$ from the maximum principle we have

$$|f(z)| \leqslant \sup_{\theta\in[0,2\pi)} |f(re^{i\theta})|,$$

where $r \geqslant |z|$ arbitrary. Since

$$(1-|z|^2)^{\alpha}|f(z)| \leqslant \sup_{|z|\leqslant r<1} (1-r^2)^{\alpha} M_{\infty}(r,f) \leqslant \|f\|_{\infty,\infty,\alpha},$$

then

$$\|F_z\| \leqslant \frac{1}{(1-|z|^2)^{\alpha}}. \quad \square$$

It has already been proven by Arévalo that the norm of the functional F_z is comparable to $\frac{1}{(1-|z|^2)^{\alpha+\frac{1}{p}}}$ in [1]. Lemma 2 provides estimates for the constants involved in the comparisons on both sides.

CONJECTURE. From the previous discussion, we conjecture that for all $0<p,q\leqslant\infty$, $0<\alpha<\infty$ and $f \in H^{p,q,\alpha}$

$$|f(z)| \leqslant \frac{\|f\|_{p,q,\alpha}}{(1-|z|^2)^{\alpha+\frac{1}{p}}}.$$

3. Proof of Theorem 1.1

Since for holomorphic $\varphi : \mathbb{D} \to \mathbb{D}$ and

$$F_z C_{\varphi} f = C_{\varphi} f(z) = f(\varphi(z)) = F_{\varphi(z)} f,$$

when $z \in \mathbb{D}$ and $f \in H^{p,q,\alpha}$ is arbitrary. Therefore, $F_z C_{\varphi} = F_{\varphi(z)}$ for every $z \in \mathbb{D}$. In terms of norms, this gives us

$$\|C_{\varphi}\| \geqslant \frac{\|F_{\varphi(z)}\|}{\|F_z\|}.$$

When $z=0$, it follows that

$$\| C_{\varphi} \| \geqslant \frac{\|F_{\varphi(0)}\|}{\|F_0\|}.$$

So, we obtain

$$\|C_{\varphi}\| \geqslant \begin{cases} \dfrac{\Gamma\left(\frac{\alpha p+\frac{p}{q}+1}{2}\right)^{\frac{2}{p}}}{\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}} \dfrac{1}{(1-|\varphi(0)|^2)^{\alpha+\frac{1}{p}}}, & 0<p,q<\infty,\ \frac{p}{q}\geqslant 1; \\ \dfrac{\Gamma\left(\frac{\alpha p+\frac{p}{q}+1}{2}\right)^{\frac{2}{p}}}{2^{\alpha+\frac{3}{p}}\Gamma(\alpha p+\frac{p}{q})^{\frac{1}{p}}} \dfrac{1}{(1-|\varphi(0)|^2)^{\alpha+\frac{1}{p}}}, & 0<p,q<\infty,\ \frac{p}{q}<1; \\ \dfrac{\Gamma\left(\frac{\alpha p+1}{2}\right)^{\frac{2}{p}}}{2^{\alpha}\Gamma(\alpha p)^{\frac{1}{p}}} \dfrac{1}{(1-|\varphi(0)|^2)^{\alpha+\frac{1}{p}}}, & 0<p<\infty,\ q=\infty; \\ \left(\dfrac{2\alpha q|\varphi(0)|}{1-|\varphi(0)|^2}+\left(\dfrac{3\sqrt{3}}{4}\right)^{\alpha q}\right)^{-\frac{1}{q}} \dfrac{1}{(1-|\varphi(0)|^2)^{\alpha}}, & p=\infty,\ 0<q<\infty; \\ \dfrac{1}{(1-|\varphi(0)|^2)^{\alpha}}, & p=\infty,\ q=\infty. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, $|\varphi(0)| = c$, $0 < p,q \leqslant \infty$, $0 < \alpha < \infty$ and $f \in H^{p,q,\alpha}$ arbitrary.

- **Case $0 < p,q < \infty$:**

To find the upper bound of the norm in this case, we use two methods, the first one is similar to the method used in [2]. From (2),

$$|\varphi(z)| \leqslant \frac{(1-c)|z|+2c}{1+c},$$

where $z \in \mathbb{D}$. For $z \in \mathbb{D}$ such that $|z| = r$ let $R = \frac{(1-c)r+2c}{1+c}$. Then from Lemma 1, we know that

$$M_p^p(r, f\circ\varphi) \leqslant \frac{R+c}{R-c} M_p^p(R, f).$$

By using the previous inequality and change of variable $R = \frac{(1-c)r+2c}{1+c}$, we find the following estimate

$$\begin{aligned} \|f\circ\varphi\|_{p,q,\alpha}^q =& 2\alpha q \int_0^1 r(1-r^2)^{\alpha q-1} M_p^q(r, f\circ\varphi)\, dr \\ \leqslant& 2\alpha q \int_0^1 r(1-r^2)^{\alpha q-1} \left(\frac{R+c}{R-c}\right)^{\frac{q}{p}} M_p^q(R, f)\, dr \\ =& \frac{(1+c)^{\alpha q}}{(1-c)^{2\alpha q}} \int_{\frac{2c}{1+c}}^1 \frac{((1+c)R-2c)(R+c)}{R(R-c)} \left(\frac{R+c}{R-c}\right)^{\frac{q}{p}-1} \cdot \\ & \cdot \left(\frac{(1+c)R+1-3c}{1+R}\right)^{\alpha q-1} R(1-R^2)^{\alpha q-1} M_p^q(R, f)\, dR. \end{aligned} \tag{6}$$

Since

$$\frac{((1+c)R-2c)(R+c)}{R(R-c)} \leqslant 1+c,$$

for $R \in \left(\frac{2c}{1+c}, 1\right)$, we have from (6)

$$\begin{aligned} \|f\circ\varphi\|_{p,q,\alpha}^q \leqslant \frac{(1+c)^{\alpha q+1}}{(1-c)^{2\alpha q}} \int_{\frac{2c}{1+c}}^1 \left(\frac{R+c}{R-c}\right)^{\frac{q}{p}-1} \left(\frac{(1+c)R+1-3c}{1+R}\right)^{\alpha q-1} \cdot \\ \cdot R(1-R^2)^{\alpha q-1} M_p^q(R, f)\, dR. \end{aligned} \tag{7}$$

To continue the estimation we need to consider the sign of $\frac{q}{p} - 1$ and $\alpha q - 1$. Let $\alpha q = a$ and $\frac{q}{p} = b$.

1. When $a \geqslant 1$, using

$$\frac{(1+c)R+1-3c}{1+R} \leqslant 1-c$$

in (7), we have

$$\|f\circ\varphi\|_{p,q,\alpha}^q \leqslant \frac{(1+c)^{a+1}}{(1-c)^{a+1}} \int_{\frac{2c}{1+c}}^1 \left(\frac{R+c}{R-c}\right)^{b-1} R(1-R^2)^{a-1} M_p^q(R, f)\, dR. \tag{8}$$

(a) For $b \leqslant 1$, we have

$$\frac{R+c}{R-c} \geqslant \frac{1+c}{1-c}. \tag{9}$$

By using (9) in (8), we obtain

$$\|f\circ\varphi\|^q_{p,q,\alpha} \leqslant \left(\frac{1+c}{1-c}\right)^{a+b} \|f\|^q_{p,q,\alpha}.$$

(b) For $b > 1$, we have

$$\frac{R+c}{R-c} \leqslant \frac{3+c}{1-c}. \tag{10}$$

After using (10) in (8), we obtain

$$\|f\circ\varphi\|^q_{p,q,\alpha} \leqslant \left(\frac{1+c}{1-c}\right)^{a+b} \left(\frac{3+c}{1+c}\right)^{b-1} \|f\|^q_{p,q,\alpha}.$$

2. In case $a < 1$ since

$$\frac{(1-c)(1+c)}{1+3c} \leqslant \frac{(1+c)R+1-3c}{1+R},$$

we have in (7)

$$\|f\circ\varphi\|^q_{p,q,\alpha} \leqslant \frac{(1+c)^{2a}}{(1-c)^{a+1}(1+3c)^{a-1}} \cdot \int_{\frac{2c}{1+c}}^1 \left(\frac{R+c}{R-c}\right)^{b-1} R(1-R^2)^{a-1} M_p^q(R,f)\,dR. \tag{11}$$

(a) If $b \leqslant 1$, after using inequality (9) in (11), we obtain

$$\|f\circ\varphi\|^q_{p,q,\alpha} \leqslant \left(\frac{1+c}{1-c}\right)^{a+b} \left(\frac{1+c}{1+3c}\right)^{a-1} \|f\|^q_{p,q,\alpha}.$$

(b) If $b > 1$, after using inequality (10) in (11), we obtain

$$\|f\circ\varphi\|^q_{p,q,\alpha} \leqslant \left(\frac{1+c}{1-c}\right)^{a+b} (1+c)^{a-b} \|f\|^q_{p,q,\alpha}.$$

From our second method, we will get a better estimate in case $a < 1$ and $b \leqslant \frac{1}{2}$. Using (1), we get

$$|\varphi(z)| \leqslant \frac{|z|+c}{1+c|z|}.$$

Let for $|z| = r < 1$, $\rho = \frac{r+c}{1+cr}$, then from Lemma 1, we get

$$\begin{aligned} \|f\circ\varphi\|_{p,q,\alpha} &= 2a \int_0^1 r(1-r^2)^{a-1} M_p^q(r, f\circ\varphi)\,dr \\ &\leqslant 2a \int_0^1 r(1-r^2)^{a-1} \left(\frac{\rho+c}{\rho-c}\right)^b M_p^q(\rho, f)\,dr. \end{aligned} \tag{12}$$

After the change of variable $\rho = \frac{r+c}{1+cr}$ in (12), we have

$$\|f\circ\varphi\|_{p,q,\alpha} \leqslant 2a\int_c^1 \frac{(\rho-c)^{1-b}(\rho+c)^b(1-c^2)^a}{(1-\rho c)^{2a+1}}(1-\rho^2)^{a-1}M_p^q(\rho,f)\,d\rho.$$

Let

$$\Phi(\rho) = \frac{(\rho-c)^{1-b}(\rho+c)^b(1-c^2)^a}{(1-\rho c)^{2a+1}}(1-\rho^2)^{a-1}M_p^q(\rho,f),$$
$$\Psi(\rho) = \rho(1-\rho^2)^{a-1}M_p^q(\rho,f).$$

If we consider function

$$h(\rho) = \frac{\Phi(\rho)}{\Psi(\rho)} = \frac{(\rho-c)^{1-b}(\rho+c)^b(1-c^2)^a}{\rho(1-\rho c)^{2a+1}}$$

when $c<\rho<1$, then

$$h'(\rho) = \frac{c(1-c^2)^a(\rho-c)^{-b}(\rho+c)^{b-1}}{\rho^2(1-\rho c)^{2a+2}}\cdot \cdot\Big(((1-2b)\rho+c)(1-c\rho)+(2a+1)\rho(\rho^2-c^2)\Big).$$

If $b\leqslant\frac{1}{2}$ then $h'(\rho)\geqslant 0$, hence

$$h(\rho)\leqslant\left(\frac{1+c}{1-c}\right)^{a+b}.$$

Since $\Phi(\rho)\leqslant\left(\frac{1+c}{1-c}\right)^{a+b}\Psi(\rho)$ for $c<\rho<1$ and $b\leqslant\frac{1}{2}$ then

$$\|f\circ\varphi\|_{p,q,\alpha}^q \leqslant 2a\int_c^1\Phi(\rho)\,d\rho \leqslant \left(\frac{1+c}{1-c}\right)^{a+b}\int_c^1\Psi(\rho)\,d\rho$$
$$\leqslant\left(\frac{1+c}{1-c}\right)^{a+b}\|f\|_{p,q,\alpha}^q.$$

- **Case $0<p<\infty, q=\infty$:**

From (2) and Lemma 1, for $z\in\mathbb{D}$ such that $|z|=r$ and $R=\frac{(1-c)r+2c}{1+c}$ we have

$$\|f\circ\varphi\|_{p,\infty,\alpha}\leqslant\sup_{0<r<1}(1-r^2)^\alpha\left(\frac{R+c}{R-c}\right)^{\frac{1}{p}}M_p(R,f)$$
$$=\left(\frac{1+c}{(1-c)^2}\right)^\alpha\sup_{\frac{2c}{1+c}\leqslant R<1}(1-R)^\alpha(1-3c+(1+c)R)^\alpha\left(\frac{R+c}{R-c}\right)^{\frac{1}{p}}M_p(R,f).$$

Since $1-3c+(1+c)R\leqslant(1-c)(1+R)$ then

$$\|f\circ\varphi\|_{p,\infty,\alpha}\leqslant\left(\frac{1+c}{1-c}\right)^{\alpha+\frac{1}{p}}\left(\frac{3+c}{1+c}\right)^{\frac{1}{p}}\|f\|_{p,\infty,\alpha}.$$

Hence,

$$\| C_\varphi \| \leqslant \left(\frac{1+c}{1-c}\right)^{\alpha+\frac{1}{p}} \left(\frac{3+c}{1+c}\right)^{\frac{1}{p}}.$$

- **Case $p=\infty, 0<q<\infty$:**

Since from the maximum principle for $0 \leqslant r < 1$ we get

$$\sup_{0 \geqslant \theta < 2\pi} |f \circ \varphi(re^{i\theta})| \leqslant \sup_{0 \geqslant \theta < 2\pi} |f(Re^{i\theta})|,$$

where $|\varphi(re^{i\theta})| \leqslant R = \frac{r+c}{1+rc}$ for all $0 \leqslant \theta < 2\pi$ from (1), then

$$\|f \circ \varphi\|^q_{\infty,q,\alpha} \leqslant 2\alpha q \int_0^1 r(1-r^2)^{\alpha q-1} M_\infty(R,f)\, dr.$$

After the change of variable $R = \frac{r+c}{1+rc}$ in the previous integral, we get

$$\|f \circ \varphi\|^q_{\infty,q,\alpha} \leqslant (1-c^2)^{\alpha q} 2\alpha q \int_c^1 \frac{R-c}{(1-Rc)^{2\alpha q+1}} (1-R^2)^{\alpha q-1} M_\infty(R,f)\, dR.$$

Since $\frac{R-c}{1-Rc} \leqslant R$ and $\frac{1}{(1-Rc)^{2\alpha q}} \leqslant \frac{1}{(1-c)^{2\alpha q}}$ when $c < R < 1$ then

$$\|f \circ \varphi\|^q_{\infty,q,\alpha} \leqslant \left(\frac{1+c}{1-c}\right)^{\alpha q} \|f\|^q_{\infty,q,\alpha}.$$

Hence, in this case

$$\| C_\varphi \| \leqslant \left(\frac{1+c}{1-c}\right)^{\alpha}.$$

- **Case $p=\infty, q=\infty$:**

From (1) and the maximum principle, we get

$$\begin{aligned}
\|f \circ \varphi\|_{\infty,\infty,\alpha} &= \sup_{0 \leqslant r<1} (1-r^2)^\alpha \sup_{0 \leqslant \theta < 2\pi} |f \circ \varphi(re^{i\theta})| \\
&\leqslant \sup_{0 \leqslant r<1} (1-r^2)^\alpha \sup_{0 \leqslant \theta < 2\pi} |f(Re^{i\theta})| \\
&= (1-c^2)^\alpha \sup_{c \leqslant R<1} \frac{1}{(1-Rc)^{2\alpha}} (1-R^2)^\alpha M_\infty(R,f) \\
&\leqslant \left(\frac{1+c}{1-c}\right)^{\alpha} \|f\|_{\infty,\infty,\alpha}.
\end{aligned}$$

Hence,

$$\| C_\varphi \| \leqslant \left(\frac{1+c}{1-c}\right)^{\alpha}.$$

This completes the proof. □

REFERENCES

[1] I. ARÉVALO, *A characterization of the inclusions between mixed norm spaces*, J. Math. Anal. Appl. **429**, 2 (2015), 942–955.

[2] I. ARÉVALO, M. D. CONTRERAS, L. RODRIGUES-PIAZZA, *Semigroups of composition operators and integral operators on mixed norm spaces*, Rev. Mat. Complut. **32**, (2019), 767–798.

[3] C. COWEN, *Linear fractional composition operators on H^2*, Integral Equations Operator Theory **11**, (1988), 151–160.

[4] C. COWEN, B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, USA, 1995.

[5] T. M. FLETT, *Lipschitz spaces of functions on the circle and the disk*, J. Math. Anal. Appl. **39**, 1 (1972), 125–158.

[6] T. M. FLETT, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38**, 3 (1972), 746–765.

[7] M. JEVTIĆ, D. VUKOTIĆ, M. ARSENOVIĆ, *Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-type Spaces*, Springer, Cham, Switzerland, 2016.

[8] A. LLINARES, D. VUKOTIĆ, *Contractive inequalities for mixed norm spaces and the Beta function*, J. Math. Anal. Appl. **509**, 1 (2022).

[9] C. LIU, *Sharp Forelli-Rudin estimates and the norm of the Bergman projection*, J. Funct. Anal. **268**, 2 (2015), 255–277.

[10] E. A. NORDGREN, *Composition operators*, Canadian J. Math. **20**, (1968), 442–449.

[11] A. SISKAKIS, *Semigroups of composition operators in Bergman spaces*, Bull. Austral. Math. Soc. **35**, 3 (1987), 397–406.

[12] H. QUEFFÉLEC, *Norms of composition operators with affine symbols*, J. Anal. **20**, (2012), 47–58.

[13] K. ZHU, *Operator Theory in Function Spaces*, second edition, American Mathematical Society, Providence, RI, USA, 2007.

(Received April 28, 2023)

Dušica Dmitrović
Faculty of Mathematics
University of Belgrade
Studentski trg 16, Belgrade, Serbia
e-mail: dusica.dmitrovic@matf.bg.ac.rs