CHARACTERIZATION-BASED APPROACH FOR CONSTRUCTION OF GOODNESS-OF-FIT TEST FOR LÉVY DISTRIBUTION

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ABSTRACT

The Lévy distribution, alongside the Normal and Cauchy distribution, is one of the only three stable distributions whose density can be obtained in a closed form. However, there are only a few specific goodness-of-fit tests for the Lévy distribution. In this paper two novel classes of goodness-of-fit tests for the Lévy distribution are proposed. Both tests are based on *V*-empirical Laplace transforms. New tests are scale free under the null hypothesis, which makes them suitable for testing the composite hypothesis. The finite sample and limiting properties of test statistics are obtained. In addition, a generalization of the recent Bhati–Kattumannil goodness-of-fit tests, the local Bahadur efficiencies are computed, and a wide power study is conducted. Both criteria clearly demonstrate the quality of the new tests. The applicability of the novel tests is demonstrated with two real-data examples.

Keywords Bahadur efficiency, Laplace transform, stable distributions, V-statistics MSC Classification 2020 62E10, 62G10, 62G20

1 Introduction

The Lévy distribution is one of the three stable distributions whose density has a closed form [39], given by

$$f(x;\lambda,\mu) = \sqrt{\frac{\lambda}{2\pi}} \frac{e^{-\frac{\lambda}{2(x-\mu)}}}{(x-\mu)^{\frac{3}{2}}}, \ x \ge \mu, \ \lambda > 0, \ \mu \in \mathbb{R}.$$
(1)

That property makes it especially attractive in the scientific community, and consequently, it has many applications (see, e.g. [10, 38, 35, 37]). Therefore, it has been of huge importance to develop methods for parameter estimation as well as appropriate goodness-of-fit (GOF) tests.

Maximum likelihood estimation of λ when $\mu = 0$ is covered in [3], while the case of both unknown parameters is addressed in [1]. However, the derivation is unclear and the numerical calculation of MLE yields different estimates. It is worth mentioning that there are results about the MLE for the parameters of stable distribution in the general case. In [28] (see also [36]), one can read that this estimation procedure is computationally demanding and it requires the maximum searching procedure to be carefully implemented. The method proposed in [20] avoids numerical optimization, but asymptotic properties seem to be unknown. The method proposed in [16] produces consistent estimates, but asymptotic properties of the estimator seem to be unknown as well. In practice, very often, the value of the location parameter can be deduced based on the nature of the phenomena, and therefore might be assumed as a known fixed value. Therefore in what follows, we assume that $\mu = 0$ and the case of unknown μ is beyond the scope of this paper. For the testing GOF to Lévy distribution, one might use classical empirical distribution function (EDF)-based GOF tests, as in [31, 18]. The usage of GOF tests for α -stable distributions for $\alpha = 0.5$ is possible as well [32]. However, as far as we know, the only specific GOF to Lévy distribution, is proposed in [6].

One of the primary goals of this paper is to fill in the existing gap in the literature with proposal of two new scale-free classes of GOF tests. New tests belong to the group of the equidistribution-characterization-based tests which recently, due to their nice properties, became very attractive. See, e.g., [24, 22, 8], for GOF tests for exponentiality, [30, 4] for GOF tests for Pareto distribution, [27] for GOF tests for logistic distributions, etc.

The common approach to assess the quality of tests is to find their power against different alternatives. This approach, for the GOF for the Lévy distribution, is used in [6]. Another approach, especially useful for a large sample comparison, is the notion of asymptotic efficiency. Bahadur asymptotic efficiency or its approximation is shown to be an attractive option when dealing with tests with non-normal limiting properties (see, e.g., [25, 9, 21]) and is recently used as one of the main criteria for the evaluation of novel proposals (see, e.g., [34, 11]). Therefore, we apply it for the novel and the Bhati–Kattumanil tests. We emphasize that this criterion has not been used before for accessing the quality of any GOF test to the Lévy distribution.

This paper is organized in the following manner. In Section 2 we revisit the test proposed in [6]. Section 3 is dedicated to the proposal of new classes of goodness-of-fit tests and their limiting properties. The asymptotic efficiency of considered tests is derived in Section 4, while the results of the empirical power study are presented in Section 5. An illustration of the usage of novel tests on real data is given in Section 6. All proofs are included in Appendix A. Appendix B contains the generalization of Bhati–Kattumanil test, while used real-data sets and their visual representations are given in Appendix C. Appendix D contains the results related to the case of the median-based estimator, while Appendix E contains empirical percentiles of the null distribution of some representatives of the proposed classes of GOF test statistics.

We will denote with $F(x; \lambda)$ the distribution function of the Lévy distribution with scale parameter λ . The density of the Lévy distribution with scale parameter λ will be denoted with $f(x; \lambda)$. The standard Lévy distribution is defined as the Lévy distribution with scale parameter $\lambda = 1$. In the sequel, the distribution function (df) and density of standard Lévy distribution will be denoted with $F_0(x)$ and $f_0(x)$ for the reasons of brevity.

2 On Bhati–Kattumanil test statistic

In [2], Ahsanullah and Nevzorov proved the following characterization of Lévy distribution.

Characterization 1. Suppose that X, Y and Z are independent and identically distributed random variables with density f defined on $(0, \infty)$. Then

Z and
$$\frac{aX + bY}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$
, $0 < a, b < \infty$

are identically distributed if and only if f is a density of Lévy distribution with arbitrary scale parameter λ .

The characterization is based upon the stability of the Lévy distribution [12]. In a view of Characterization 1, for a = b = 1, Bhati and Kattumannil in [6] proposed the test statistic

$$T_n^* = \int_{\mathbb{R}^+} \left(\frac{1}{\binom{n}{2}} \sum_{j < i} I\left\{\frac{X_i + X_j}{4} \le t\right\} - F_n(t)\right) dF_n(t),$$

which is an integrated difference between U-empirical df of X and $\frac{X+Y}{4}$. This statistic is a hybrid U-statistic, asymptotically equivalent to the non-degenerate U-statistic

$$T_n = \frac{1}{\binom{n}{3}} \sum_{k < j < i} I\left\{\frac{X_i + X_j}{4} \le X_k\right\} - \frac{1}{2}.$$

They showed that, under H_0 , the distribution $\sqrt{n}T_n$ converged to a centred Gaussian distribution with variance equal to

$$\sigma_T^2 = Var\Big(\int_0^\infty 2\Big(1 - F_0\Big(\frac{X+y}{4}\Big)\Big)f_0(y)dy + F_0(X)\Big).$$
(2)

Since they were not being able to calculate asymptotic variance, they proceeded with considering empirical jackknife and jackknife-adjusted versions of T_n . The fact that the test statistic is scale free under the null hypothesis, as it will

be shown later, significantly simplifies testing procedures and the usage of the aforementioned resampling procedures might be skipped.

Our numerical calculation yields $\sigma_T^2 = 0.0235051$.

In the rest of this paper, we will denote the statistic T_n with $\bar{I}_n^{[1,1]}$, and its generalization for arbitrary a, b > 0, presented in Appendix B, with $\bar{I}_n^{[a,b]}$. Since the properties of $\bar{I}_n^{[a,b]}$ aren't significantly different than those of the a = b = 1, we present them in Appendix B.

3 New classes of goodness-of-fit tests

Equality in distribution of two random variables can also be expressed through the equality of their Laplace transforms. The test statistic might be constructed as a function of the difference among corresponding U- or V-empirical Laplace transforms. This approach has been used for the first time in [23], and was further explored in [8, 9]. Taking into account the discussions from mentioned papers, in a view of Characterization 1, we propose new classes of test statistics $\mathcal{J} = \{J_{n,a}, a > 0\}$ and $\mathcal{R} = \{R_{n,a}, a > 0\}$, where

$$J_{n,a} = \sup_{t>0} \left| \left(\frac{1}{n^2} \sum_{i,j} e^{-\frac{t(Y_i + Y_j)}{4}} - \frac{1}{n} \sum_i e^{-tY_i} \right) e^{-at} t^{\frac{3}{2}} \right|$$

$$| \left(\frac{1}{n^2} \sum_{i,j} e^{\frac{Y_i + Y_j}{4}} - \frac{1}{n^2} \sum_i e^{-tY_i} \right) e^{-at} t^{\frac{3}{2}} |$$
(3)

$$= \sup_{t \in [0,1]} \left| \left(\frac{1}{n^2} \sum_{i,j} t^{-4} - \frac{1}{n} \sum_i t^{i} \right) t^a \left(-\log t \right)^2 \right|,$$

$$R_{n,a} = \int_{\mathbb{R}^+} \left(\frac{1}{n} \sum_i e^{-tY_i} - \frac{1}{n^2} \sum_{i,j} e^{-\frac{t(Y_i + Y_j)}{4}} \right) e^{-at} t^{\frac{3}{2}} dt \qquad (4)$$

$$= \frac{3\sqrt{\pi}}{4n^2} \sum_{i,j} \left(\frac{1}{\left(a + \frac{Y_i + Y_j}{4}\right)^{\frac{5}{2}}} - \frac{1}{2\left(a + Y_i\right)^{\frac{5}{2}}} - \frac{1}{2\left(a + Y_j\right)^{\frac{5}{2}}} \right),$$

and $Y_k = \frac{X_k}{\hat{\lambda}}$ and $\hat{\lambda}$ is MLE of λ , given by

$$\hat{\lambda} = \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{X_k}}.$$
(5)

Note that one might also opt for the median-based estimator (MBE):

$$\hat{A}_{MBE} := 2(\mathrm{erf}^{-1}(1/2))^2 \widetilde{x}_{2}$$

 ∞

where \widetilde{x} is the sample median, and where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt$$

denotes the complementary error function. The power study for this approach is presented in Appendix D. We will assume $\hat{\lambda}$ is the MLE in the rest of this paper.

Here the function $e^{-at}t^{3/2}$ plays a role of weight function. Therefore, test statistics might be modified with the selection of another weight function. Since, under H_0 , the values of test statistics should be small, we take large values of $J_{n,a}$ and $|R_{n,a}|$ to be significant.

In the next two theorems, we present limiting distributions of $\sqrt{n}J_{n,a}$ and $\sqrt{n}R_{n,a}$ under H_0 .

Theorem 1. Let $a \ge 1$ and X_1, X_2, \ldots, X_n be i.i.d random variables distributed according to the Lévy law with scale parameter λ . Then the following holds:

$$\sqrt{n}J_{n,a} \xrightarrow{D} \sup_{t \in [0,1]} |\xi(t)|,$$

where $\xi(t)$ is a centred Gaussian random process, having the following covariance function:

$$K(s,t) = s^{a}t^{a}(-\log(s))^{3/2}(-\log(t))^{3/2}\left(-e^{-\sqrt{2}\left(\sqrt{-\log(s)} + \sqrt{(-\log(t))}\right)} - 2e^{-\sqrt{-2}\left(\log(s) - \frac{1}{4}\log(t)\right)} + \sqrt{-\frac{\log(t)}{2}} - 2e^{-\sqrt{2}\left(-\log(t) - \frac{1}{4}\log(s)\right)} + \sqrt{-\frac{\log(s)}{2}} + 4e^{-\frac{\sqrt{-\log(s)} + \sqrt{-\log(s)} + \sqrt{-\log(t)}}{\sqrt{2}}} + e^{\sqrt{-2\log(st)}}\right).$$

Empirical 95th percentiles of $\sqrt{n}J_{n,a}$, presented in Table 11 and Table 12 in Appendix E are in concordance with Theorem 1.

Remark 1. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables distributed as $X \in F_A$ and let F_A be a fixed alternative distribution such that $E\left(\frac{1}{X}\right)^2 < \infty$. Denote with $\zeta = \frac{1}{E\left(\frac{1}{X}\right)}$. The law of large numbers along with the

continuous mapping theorem gives us $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}} \xrightarrow{P} \zeta$. Next, applying the law of large numbers for U-and V-statistics with estimated parameters [13], and similar arguments as in the proof of Lemma 1 we have that $J_n \xrightarrow{P} \sup_{t \in [0,1]} |(E(t^{\frac{X_1}{\zeta}}) - E(t^{\frac{X_1+X_2}{4\zeta}}))t^a(-\log t)^{\frac{3}{2}}|$ which is equal to 0 if and only if $\frac{X_1}{\zeta}$ and $\frac{X_1+X_2}{4\zeta}$ are equally distributed, i.e. iff the null hypothesis holds. From this, consistency of $J_{n,a}$ follows.

Theorem 2. Let $a \ge 1$ and X_1, X_2, \ldots, X_n be i.i.d random variables distributed according to the Lévy law with scale parameter λ . Then, for every a > 0, the asymptotic distribution of $\sqrt{nR_{n,a}}$ as $n \to \infty$ is normal $\mathcal{N}(0, \sigma_R^2(a))$ where $\sigma_R^2(a) = 4E\zeta(X;a)^2$ and $\zeta(x;a)$ is defined in (15).

The values of $\sigma_R^2(a)$ cannot be calculated analytically. However, it is possible to calculate them numerically. Some values of $\sigma_R^2(a)$ are presented in Table 1. Therefore, the testing procedure can also be done using the standardized test statistic

$$\widetilde{R}_{n,a} = \sqrt{n} \frac{R_{n,a}}{\sigma_R(a)}$$

which, for large samples, under H_0 , can be approximated with standard normal distribution.

Table 1: Values of $\sigma_R^2(a)$ for different values of a.

$\sigma_R^2(0.2)$	$\sigma_R^2(0.5)$	$\sigma_R^2(1)$	$\sigma_R^2(2)$	$\sigma_R^2(5)$
4.58804	0.2672024	0.02068868	0.001194688	$1.925016 \cdot 10^{-5}$

Empirical 95th percentiles of $\sqrt{n}|R_{n,a}|$, presented in Table 13 and Table 14 in Appendix E, are in concordance with Theorem 2.

4 Asymptotic efficiency

In recent times, the Bahadur efficiency has become a very popular tool for stochastic comparison of test performance in the large sample case. In this section we make a brief review of Bahadur theory. For more details, we refer to [25].

Let $\mathcal{G} = \{g(x; \theta), \theta > 0\}$ be a family of alternatives density functions, such that g(x; 0) has the Lévy distribution with arbitrary scale parameter, and $\int_{\mathbb{R}^+} \frac{1}{x^2} g(x; \theta) < \infty$ for θ in the neighbourhood of 0, and some additional regularity conditions for U-statistics with non-degenerate kernels hold [26, 21]. Let also $\{T_n\}$ and $\{V_n\}$ be two sequences of test statistic that we want to compare.

Then for any alternative distribution from \mathcal{G} the relative Bahadur efficiency of the $\{T_n\}$ with respect to $\{V_n\}$ can be expressed as

$$e_{(T,V)}(\theta) = \frac{c_T(\theta)}{c_V(\theta)},$$

where $c_T(\theta)$ and $c_V(\theta)$ are the Bahadur exact slopes, functions proportional to the exponential rate of decrease of each test size when the sample size increases. It is usually assumed that θ belongs to the neighbourhood of 0, and in such cases, we refer to the local relative Bahadur efficiency of considered sequences of test statistics.

It is well known that for the Bahadur slope function Bahadur–Ragavacharri inequality holds [33], that is

$$c_T(\theta) \le 2K(\theta)$$

where $K(\theta)$ is the minimal Kullback–Leibler distance from the alternative to the class of null hypotheses, i.e. in the case of our null hypothesis

$$K(\theta) = \inf_{\lambda > 0} K(\theta; \lambda) = \inf_{\lambda > 0} \int_{\mathbb{R}^+} \log \left(\frac{g(x; \theta)}{f(x; \lambda)} \right) g(x; \theta) dx.$$

This justifies the definition of the local absolute Bahadur efficiency by

$$eff(T) = \lim_{\theta \to 0} \frac{c_T(\theta)}{2K(\theta)}.$$
(6)

If the sequence $\{T_n\}$ of test statistics under the alternative converges in probability to some finite function $b(\theta) > 0$ and the limit

$$\lim_{n \leftarrow \infty} n^{-1} \log P_{H_0}(T_n \ge t) = -f_{LD}(t)$$

exists for any t in an open interval I, on which f_{LD} is continuous and $\{b(\theta), \theta > 0\} \subset I$ then the Bahadur exact slope is equal to

$$c_T(\theta) = 2f_{LD}(b(\theta)). \tag{7}$$

However, in many cases, the calculation of the large deviation function, and consequently the Bahadur slope, turns out to be almost an insurmountable obstacle.

If the function (4) cannot be calculated for θ approaching zero, instead of calculating Bahadur slope we could calculate approximate Bahadur slope $c_T^*(\theta)$ which usually locally coincides with the exact one. For the calculation of the approximate slope, we do not need the tail behaviour of d.f. of statistics T_n but the tail behaviour of its limiting distribution, which is often easier to calculate. In particular, if the limiting distribution function of T_n , under H_0 , is

 F_T , whose tail behaviour is given by $\log(1 - F_T(t)) = -\frac{a_T^* t^2}{2}(1 + o(1))$, where a_T is the positive real number and $o(1) \to 0$ when $t \to \infty$, and the limit in probability of $\frac{T_n}{\sqrt{n}}$ is $b_T^*(\theta) > 0$, then

$$c_T^*(\theta) = a_T^*(b_T^*(\theta))^2.$$

In addition, the local (approximate) slope of likelihood ratio tests is equal to $2K(\theta)$ [5], therefore it is reasonable to approximate (6) by replacing $c_T(\theta)$ with $c_T^*(\theta)$.

In the next theorem, we provide the behaviour of $2K(\theta)$ when θ approaches zero.

Theorem 3. For a given alternative density $g(x; \theta)$ whose distribution belongs to \mathcal{G} , such that g(x; 0) is given by (1), *it holds*

$$2K(\theta) = \left(\sqrt{\frac{2\pi}{\lambda}} \int_{\mathbb{R}^+} (g'(x;\theta))^2 e^{-\frac{\lambda}{2x}} x^{-\frac{3}{2}} dx - \frac{\lambda}{2} \left(\int_{\mathbb{R}^+} \frac{g'(x;\theta)}{x} dx\right)^2 \right) \cdot \theta^2 + o(\theta^2), \ \theta \to 0.$$

In the following theorems, we present Bahadur exact and approximate slopes of Bhati-Kattumanil statistic and our statistics, respectively.

Theorem 4. For an alternative $g(x; \theta)$ from \mathcal{G} , the Bahadur exact slope of the statistic $I_n^{[1,1]}$ is

$$c_I(\theta) = \frac{\left(3\int_{\mathbb{R}^+} \varphi(x)g'_{\theta}(x;0)dx\right)^2}{\sigma_T^2} \cdot \theta^2 + o(\theta^2), \theta \to 0,$$

where

$$\varphi(x) = \int_{0}^{\infty} 2\left(1 - F\left(\frac{x+y}{4};\lambda\right)\right) f(y;\lambda) dy + F(x;\lambda).$$

Proof of Theorem 4. The proof follows directly from [26, Theorem 3].

Theorem 5. For an alternative $g(x; \theta)$ from \mathcal{G} , the Bahadur approximate slope of the statistic $J_{n,a}$ is equal to

$$c_J(\theta) = \frac{\sup_{t \in [0,1]} \left(2|\int_{\mathbb{R}^+} \psi(x;t,a) g'_{\theta}(x;0) dx| \right)^2}{\sup_{t \in [0,1]} \sigma^2(t)} \cdot \theta^2 + o(\theta^2),$$
(8)

where $\sigma^2(t) = \sup_{t \in [0,1]} K(t,t)$, and

$$\psi(x;t,a) = \frac{1}{2}t^a(-\log(t))^{3/2} \Big(-2t^{\frac{x}{4}}e^{-\frac{\sqrt{-\log(t)}}{\sqrt{2}}} + t^x + e^{-\sqrt{2}\sqrt{-\log(t)}}\Big).$$

Theorem 6. For an alternative $g(x; \theta)$ from \mathcal{G} , the Bahadur exact slope of the statistic $R_{n,a}$ is

$$c_R(\theta) = \frac{\left(2\int_{\mathbb{R}^+} \zeta(x)g'_{\theta}(x;0)dx\right)^2}{\sigma_R^2(a)} \cdot \theta^2 + o(\theta^2), \theta \to 0,$$

where ζ is the first projection of the symmetric kernel Z given by

$$\zeta(x;a) = E(Z(X_1, X_2; a) | X_1 = x)$$

The expression for ζ *is cumbersome and the exact form can be found in Appendix, (see 15).*

We consider the following classes of alternatives that belong to family \mathcal{G} :

• a mixture of the standard Lévy distribution and the Lévy distribution with scale parameter $\lambda \neq 1$, with density:

$$g_1^{[\lambda]}(x;\theta) = (1-\theta)f_0(x) + \frac{\theta}{\lambda}f_0\left(\frac{x}{\lambda}\right), \ x > 0, \ \theta \in (0,1);$$

• a Lehmann alternative with density

$$g_2(x;\theta) = (1+\theta)F_0(x)^{\theta}f_0(x), \ x > 0, \ \theta > 0$$

• a contamination alternative with g_2 , and parameter β with density

$$g_3^{[\beta]}(x;\theta) = (1-\theta)f_0(x) + \theta\beta F_0^{\beta-1}(x)f_0(x), \ x > 0, \ \theta \in (0,1), \ \beta > 0;$$

• a first Ley-Paindaveine alternative [17] with density

$$g_4(x;\theta) = (1 + \theta F_0(x))f_0(x)e^{-\theta(1 - F_0(x))}, x > 0, \theta > 0;$$

• a second Ley-Paindaveine alternative [17] with density

$$g_5(x;\theta) = f_0(x)(1 - \theta\pi \cos\left(\pi F_0(x)\right)), \ x > 0, \ \theta \in [0, \pi^{-1}].$$

In what follows we present a calculation of the local approximate Bahadur efficiency of J_1 and alternative $g_2(x;\theta)$, while results for all considered statistics and alternatives are presented in Table 2.

From Theorem 3 we obtain

$$2K(\theta) = \left(\int_{\mathbb{R}^+} \frac{e^{-\frac{1}{2x}} \left(\log\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{x}}\right)\right) + 1\right)^2}{\sqrt{2\pi}x^{3/2}} dx - \frac{1}{2} \int_{\mathbb{R}^+} \frac{e^{-\frac{1}{2x}} \left(\log\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{x}}\right)\right) + 1\right)}{\sqrt{2\pi}x^{5/2}} dx\right) \cdot \theta^2 + o(\theta^2)$$
$$= 0.0233005\theta^2 + o(\theta^2), \ \theta \to 0.$$

Next, from Theorem 5, it follows that

$$c_{J}(\theta) = \frac{4\sup_{t \in [0,1]} A(t)}{\sup_{t \in [0,1]} \sigma^{2}(t)} \theta^{2} + o(\theta^{2}) = \frac{4\sup_{t \in [0,1]} \left(\left| \int_{\mathbb{R}^{+}} \psi(x;t,a)g_{\theta}'(x;0)dx \right| \right)^{2}}{\sup_{t \in [0,1]} \sigma^{2}(t)} \cdot \theta^{2} + o(\theta^{2}).$$

We have that

$$\sup_{t \in [0,1]} A(t) = \sup_{t \in [0,1]} \left(\int_{\mathbb{R}^+} t^a (-\log(t))^{3/2} \left(\log\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{y}}\right) \right) + 1 \right) \times \frac{e^{-\sqrt{2}\sqrt{-\log(t)} - \frac{1}{2y}} \left(-2t^{y/4} e^{\frac{\sqrt{-\log(t)}}{\sqrt{2}}} + t^y e^{\sqrt{2}\sqrt{-\log(t)}} + 1 \right)}{2\sqrt{2\pi}y^{3/2}} dy \right)^2 \approx 0.0000149667,$$

We highlight that the maximum of the function A(t) (presented in Figure 1) is calculated numerically.



Figure 2: Plot of $\sigma^2(t)$.

Further, $\sup_{t \in [0,1]} \sigma^2(t)$ becomes

$$\sup_{t \in [0,1]} \left(-t^2 \left(-4e^{-\frac{2\sqrt{-\log(t^{5/4})} + \sqrt{-\log(t)}}{\sqrt{2}}} + 4e^{-\frac{\sqrt{-\log(t^2)} + 2\sqrt{-\log(t)}}{\sqrt{2}}} + e^{-\sqrt{-2\log(t^2)}} - e^{-2\sqrt{-2\log(t)}} \right) \log^3(t)$$

$$\approx 0.00388889.$$

and the calculation yields $eff(J_1) \approx 0.66$. The function $\sigma^2(t)$ is presented in Figure 2.

The results of our study are displayed in Table 2. The tuning parameter *a* significantly affects the efficiency of $R_{n,a}$ and $J_{n,a}$. In all considered cases, the Bahadur efficiency of $J_{n,a}$ is a decreasing function of *a*. This is not the case for the statistic $R_{n,a}$. However, it is notable that the maximal efficiency is attained in the neighbourhood of a = 1. The values of the local approximate Bahadur relative efficiencies for the generalized Bhati–Kattumanil statistic are given in Appendix B. We can see that the new statistics outperform the Bhati–Kattumanil one, and that the statistic $R_{n,a}$ dominates the other two in terms of local approximate Bahadur efficiencies.

Remark 2. It can be shown that for the test statistic $I^{[1,1]}$ considered in this paper approximate and exact slopes coincide locally, and consequently, the approximate and exact Bahadur relative efficiencies w.r.t. LR test locally coincide because the approximate Bahadur slope for LR test equals to $2K(\theta)$, as we have noted earlier.

5 Power study

In this section, we explore finite sample properties of considered test statistics. In particular, we estimate the power of the tests, when the level of significance is $\alpha = 0.05$, using Monte Carlo method with N = 10000 replications. The goal of this section is to compare JEL and AJEL approaches from [6] to the classical approach and to determine the empirical powers of new tests. The *p*-values are obtained utilizing the Monte Carlo approach.

The supremum in the calculation of the $J_{n,a}$ is obtained using grid search on 1000 equidistant points on [0, 1].

Other tests considered in the power study are:

	$g_1^{[10]}$	g_2	$g_{3}^{[3]}$	g_4	g_5
$I^{[1,1]}$	0.59	0.54	0.73	0.53	0.41
J_1	0.91	0.66	0.79	0.68	0.69
J_2	0.81	0.54	0.71	0.54	0.49
J_5	0.56	0.36	0.52	0.35	0.25
J_{10}	0.35	0.24	0.37	0.23	0.13
$R_{0.2}$	0.53	0.79	0.61	0.80	0.86
$R_{0.5}$	0.80	0.86	0.82	0.90	0.97
R_1	0.94	0.81	0.89	0.84	0.87
R_2	0.93	0.69	0.84	0.70	0.65
R_5	0.70	0.48	0.66	0.46	0.35

Table 2: Local approximate Bahadur relative efficiencies of $I^{[1,1]}$, J_a and R_a with respect to LR test

• Lilliefors-corrected Kolmogorov-Smirnov (KS) test statistic:

$$KS_n = \sup_{x \in \mathbb{R}^+} |F_n(x) - F(x; \hat{\lambda})|;$$

• Lilliefors-corrected Cramer-von Mises (CVM) test statistic:

$$CVM_n^2 = \int_{\mathbb{R}^+} (F_n(x) - F(x;\hat{\lambda}))^2 dF(x;\hat{\lambda});$$

• Lilliefors-corrected Anderson–Darling (AD) test statistic:

$$AD_n^2 = n \int\limits_{\mathbb{R}^+} \frac{(F_n(x) - F(x;\hat{\lambda}))^2}{F(x;\hat{\lambda})(1 - F(x;\hat{\lambda}))} dF(x;\hat{\lambda});$$

• N_1^a test statistic, considered in [32]:

$$N_1^a = \sqrt{n} \cdot \frac{\hat{\sigma}_x^2(5\%, 25\%) - \hat{\sigma}_x^2(75\%, 95\%)}{\hat{\sigma}_x^2(5\%, 95\%)}$$

• N_1^b test statistic, considered in [32]:

$$N_1^b = \sqrt{n} \cdot \frac{2.00 \cdot \hat{\sigma}_x^2(5\%, 25\%) - 1.01 \cdot \hat{\sigma}_x^2(75\%, 95\%)}{\hat{\sigma}_x^2(5\%, 95\%)}$$

where $\hat{\sigma}_x^2$ denotes the sample quantile conditional variance estimator [32]:

$$\hat{\sigma}_x^2(a,b) = \frac{1}{[nb] - [na]} \sum_{i=[na]+1}^{[nb]} \left(X_{(i)} - \hat{\mu}_X(a,b) \right)^2,$$

and $X_{(i)}$ is the ith-order statistic, and $\hat{\mu}_X(a,b) = \frac{1}{[nb]-[na]} \sum_{i=[na]+1}^{[nb]} X_{(i)}$ is the conditional sample mean.

In the case of Lilliefors-corrected classical tests large values are taken to be significant, while in the case of both small and large values of N_1^a and N_1^b , the null hypothesis is rejected.

We consider the following classes of alternative distributions:

• Burr distribution, denoted by Burr(a, b, c), with a density

$$g_B(x;a,b,c) = cb \frac{\left(\frac{x}{a}\right)^{b-1}}{a\left(1 + \left(\frac{x}{a}\right)^b\right)^{c+1}}, \ x > 0, \ a > 0, \ b > 0, \ c > 0;$$

• Chen distribution, denoted by $Chen(\nu, \lambda)$, with a density

$$g_C(x;\nu,\lambda) = \nu\lambda x^{\lambda-1} e^{\nu(1-e^{x^{\lambda}})+x^{\lambda}}, \ x > 0, \ \nu > 0, \ \lambda > 0;$$

• Fréchet distribution, denoted by FR(a, b), with a density

$$g_{FR}(x;a,b) = \frac{a}{b} \left(\frac{x}{b}\right)^{-(a+1)} \exp\left(-\left(\frac{x}{b}\right)^{-a}\right), \ x > 0, \ a > 0, \ b > 0;$$

• Gamma distribution, denoted by $\Gamma(a, b)$, with a density

$$g_{\Gamma}(x;a,b) = \frac{x^{a-1}b^a e^{-bx}}{\Gamma(a)}, \ x > 0, \ a > 0, \ b > 0;$$

• log-logistic distribution, denoted by LL(*a*, *b*), with a density

$$g_{LL}(x;a,b) = \frac{a(\frac{x}{b})^a}{x[1+a(\frac{x}{b})^a]^2}, \ x > 0, \ a > 0, \ b > 0;$$

• log-normal distribution, denoted by LN(a, b), with a density

$$g_{LN}(x;a,b) = \frac{1}{\sqrt{(2\pi)bx}} e^{\frac{-((\log x - a)^2}{2b^2}}, \ x > 0, \ a \in \mathbb{R}, \ b > 0;$$

• χ^2 distribution, denoted by χ^2_n , with a density

$$g_{\chi^2}(x;n) = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \ x > 0, \ n \in \mathbb{N};$$

• half-normal distribution, denoted by HN(a, b), with a density

$$g_{HN}(x;a,b) = \frac{e^{-\frac{(x-a)^2}{2b^2}}}{\sqrt{2\pi b^2}} + \frac{e^{-\frac{(x+a)^2}{2b^2}}}{\sqrt{2\pi b^2}}, \ x > a, \ a \in \mathbb{R}, \ b > 0;$$

• shifted log-gamma distribution, denoted by LG(a, b), with a density

$$g_{LG}(x;a,b) = \frac{b^a}{\Gamma(a)} \frac{(\log(x+1))^{a-1}}{(x+1)^{b+1}}, \ x > 0, \ a > 0, \ b > 0;$$

• Weibull distribution, denoted by W(a, b) with a density

$$g_W(x;a,b) = \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} e^{\left(-\frac{x}{b}\right)^a}, \ x > 0, \ a > 0, \ b > 0;$$

The majority of these alternatives are considered in [6].

N_{i}^{h}	0.05	0.05	0.05	0.42	0.38	0.59	0.63	0.79	0.87	0.03	0.04	0.04	0.05	0.22	0.10	0.18	0.05	0.05	0.05	0.72	0.76	0.95	0.97	0.99	1	0.04	0.04	0.04	0.10	0.46	0.19	0.34
N_1^a	0.05	0.06	0.05	0.18	0.30	0.57	0.73	0.73	0.86	0.22	0.06	0.07	0.24	0.15	0.09	0.10	0.05	0.05	0.05	0.27	0.74	0.98	1	1	1	0.65	0.17	0.17	0.73	0.36	0.21	0.22
AD	0.05	0.05	0.05	1	0.91	0.98	0.97	1	1	0.99	0.99	0.55	0.92	0.32	0.48	0.67	0.05	0.05	0.05	1	1	1	1	1	1	1	1	0.84	1	0.65	0.76	0.96
CVM	0.05	0.05	0.05	1	0.88	0.96	0.95	1	1	0.99	0.99	0.56	0.93	0.30	0.48	0.68	0.05	0.05	0.05	1	1	1	1	1	1	1	1	0.84	1	0.55	0.75	0.95
KS	0.06	0.06	0.05	1	0.86	0.96	0.96	1	1	0.99	0.98	0.53	0.91	0.27	0.47	0.59	0.05	0.05	0.05	1	1	1	1	1	1	1	1	0.81	1	0.51	0.72	0.90
AJEL	0.04	0.04	0.04	0.21	0.24	0.39	0.42	0.76	0.77	0.04	0.05	0.04	0.05	0.07	0.05	0.09	0.04	0.04	0.05	0.35	0.57	0.72	0.73	0.98	0.97	0.05	0.05	0.04	0.06	0.11	0.05	0.19
JEL /	0.06	0.05	0.05	0.33	0.29	0.52	0.53	0.83	0.79	0.06	0.07	0.06	0.07	0.08	0.06	0.11	0.05	0.05	0.05	0.41	0.62	0.67	0.76	0.99	0.99	0.06	0.06	0.05	0.08	0.13	0.06	0.21
R _E	0.05	0.05	0.05	0.59	66 .0	0.97	0.85	1	1	0.43	0.42	0.05	0.22	0.53	0.11	0.69	0.05	0.05	0.05	0.72	1	0.99	0.91	1	1	0.59	0.60	0.10	0.48	0.84	0.12	0.94
B, B,	0.05	0.05	0.05	0.60	0.95	0.87	0.65	1	1	0.48	0.49	0.10	0.33	0.45	0.08	0.77	0.05	0.05	0.05	0.68	1	0.95	0.71	1	1	0.57	0.57	0.29	0.58	0.68	0.09	0.96
	0.05	0.05	0.05	0.64	0.82	0.69	0.45	1	0.98	0.56	0.56	0.20	0.47	0.32	0.11	0.74	0.05	0.05	0.05	0.73	0.97	0.82	0.46	1	1	0.66	0.64	0.51	0.69	0.54	0.22	0.95
Ro F	0.05	0.05	0.05	0.98	0.63	0.48	0.35	0.98	0.93	0.97	0.97	0.39	0.85	0.23	0.28	0.66	0.05	0.05	0.04	0.81	0.83	0.53	0.32	1	0.98	0.74	0.74	0.69	0.80	0.33	0.48	0.91
Ro o	0.05	0.05	0.05	0.98	0.37	0.36	0.43	0.87	0.76	0.98	0.98	0.58	0.93	0.14	0.51	0.49	0.05	0.05	0.05	1	0.51	0.36	0.55	0.95	0.84	1	1	0.86	1	0.18	0.77	0.80
<u>J</u> 10	0.05	0.05	0.05	0.53	0.99	0.99	0.92	1	1	0.44	0.47	0.22	0.46	0.52	0.14	0.60	0.05	0.05	0.05	0.66	1	1	0.95	1	1	0.58	0.59	0.52	0.64	0.81	0.20	0.89
JE	0.05	0.05	0.05	0.43	0.98	0.96	0.83	1	1	0.44	0.47	0.22	0.47	0.47	0.10	0.70	0.05	0.05	0.05	0.58	1	0.99	0.89	1	1	0.58	0.59	0.53	0.64	0.77	0.11	0.93
. <i>I</i> ,	0.04	0.05	0.05	0.43	0.89	0.83	0.60	1	1	0.45	0.47	0.22	0.46	0.35	0.07	0.70	0.05	0.05	0.05	0.56	0.99	0.94	0.68	1	1	0.59	0.58	0.52	0.64	0.58	0.13	0.93
Π,	0.05	0.05	0.05	0.55	0.71	0.61	0.41	1	0.98	0.44	0.47	0.22	0.46	0.25	0.13	0.64	0.05	0.05	0.05	0.67	0.93	0.76	0.48	-	1	0.59	0.58	0.53	0.64	0.36	0.30	0.89
$\overline{I}^{[1,1]}$	0.05	0.04	0.05	0	0.09	1	1	1	1	0.16	0	0.06	0.06	0.56	0.24	0.76	0.05	0.05	0.05	0	1	1	1	1	1	0.27	0.01	0.07	0.51	0.88	0.43	0.95
6	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
Distribution	Lévy(0, 0.5)	Levy(0, 1)	Lévy(0, 2)	Burr(1.5, 0.5, 0.5)	LN(0, 1)	$\chi^2(3)$	HN(0,1)	$\Gamma(3,2)$	W(2, 1)	$\Gamma(0.4,2)$	W(0.4,2)	LN(0,2)	Chen(2, 0.4)	LG(7,2)	LL(1, 2)	FR(1, 1)	Lévy(0, 0.5)	Levy(0, 1)	Lévy(0, 2)	Burr(1.5, 0.5, 0.5)	LN(0, 1)	$\chi^2(3)$	HN(0,1)	$\Gamma(3,2)$	W(2, 1)	$\Gamma(0.4,2)$	W(0.4, 2)	LN(0,2)	Chen(2, 0.4)	LG(7, 2)	LL(1, 2)	FR(1, 1)

ML estimate
powers -
empirical
of
Comparison
Table 3:

Results are presented in Table 3. For the sake of brevity, sample sizes are dropped from labels whenever they can be clearly determined. When the tests are compared, it can be seen that no new test is preferable among the others. That is in concordance with [14], which asserts that the global power function of any nonparametric test is flat on balls of alternatives except for alternatives coming from a finite-dimensional subspace.

From Table 3, it can be seen that JEL and AJEL approaches, proposed in [6], are less powerful than classical, whenever the testing is utilized via the original version of $|I^{[1,1]}|$. It is also notable that the power of R_a is significantly affected by the value of tuning parameter a and alternative distribution. Having all results in mind we recommend the value in the interval [0.5, 1]. The behaviour of J_a is less sensitive to the change of parameter a. In almost all cases both R_a and J_a dominate JEL and AJEL competitors. It can be concluded that novel tests exhibit better performance when compared with the tests N_1^a and N_1^b , proposed in [32]. When compared to EDF-based tests, in some cases novel tests show better performance, while in other cases they are comparable.

6 Real data examples

In this section, we apply the novel tests presented in this paper on two real data sets considered in [6]. The data sets and their visual representations are provided in Appendix C.

The first one (Rainfall) contains the weighted rainfall data for the month of January in India. Although there is no objective reason for modelling data with such a shape with the Lévy distribution, in [6], authors concluded that the data follows the Lévy distribution [6, p. 10]. However, all tests we consider report p-values smaller than 0.05, which clearly implies that Lévy distribution is not a justified choice.

The second data set (Hillside) consists of the well yields near Bel Air, Hartford county, Maryland. The *p*-values are presented in Table 4 (see also Table 9). From the Figure 4 presented in Appendix C, it can be deduced that the empirical density of the Hillside data is, among the distributions studied in the simulation study, closest to LL(1, 2). Having in mind $R_{0,2}$ that is quite powerful against this alternative, we cannot conclude that the Lévy distribution is the appropriate model for the Hillside data.

Table 4	: p-value	es of nov	el tests -	ML esti	mate
	$R_{0.2}$	$R_{0.5}$	R_1	R_2	R_5
Rainfall Hillside	0.014 0.004	0 0.006	0 0.024	0 0.106	0 0.494
	J_1	J_2	J_5	J_{10}	
Rainfall Hillside	0 0.021	0 0.07	0 0.281	0 0.622	

7 Concluding remarks

In this paper, we proposed two new goodness-of-fit tests for the Lévy distribution with arbitrary scale parameter and a generalization of an existing one. The asymptotic distributions of the proposed tests were derived and the local approximate Bahadur efficiencies of the proposed tests and the generalized Bhati–Kattumanil test were compared. Obtained empirical powers also clearly indicate the dominance over other specific goodness-of-fit tests for the Lévy distribution.

We end our work by identifying some open research questions. From the results of our empirical study presented in Section 5 and Appendix D, we note that the test's powers are sensitive to the choice of the estimator of the scale parameter. Therefore it is of interest to further analyse the small sample behaviour of our tests under different estimators, not included in the study. In addition, a prominent direction of further research would be to look for the adaptation of proposed tests when both location and scale parameters are unknown. This case is even more challenging for developing asymptotic properties of test statistics under the null and fixed alternative as well. It would be also interesting to look for tests' behaviour under contiguous alternatives.

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Declaration of interests

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Appendix A – Proofs

Proof of Theorem 1. It is easy to see that, under H_0 , the distribution of $J_{n,a}$ does not depend on λ which justifies the usage of the introduced class for testing the composite hypothesis. Therefore, when deriving the limiting distribution we may suppose that $\lambda = 1$. It should be noted that the statistic (3) can be represented as

$$J_{n,a} = \sup_{t \in [0,1]} |V_n(t;a,\hat{\lambda})|,$$

where, for each t and a,

$$V_n(t;a,\hat{\lambda}) = \frac{1}{n^2} \sum_{i_1,i_2}^n \Psi(X_{i_1}, X_{i_2}; t, a, \hat{\lambda}),$$

is a V-statistic with the estimated parameter λ , with symmetric kernel $\Psi(\cdot; t, a)$. Applying the Taylor expansion, we get

$$\sqrt{n}V_n(t;a,\hat{\lambda}) = \sqrt{n}V_n(t;a,1) + \sqrt{n}(\hat{\lambda}-1)\frac{\partial V_n(t;a,\nu)}{\partial \nu}\Big|_{\nu=1} + R_n(t),$$
(9)

where

$$R_n(t) = \frac{\sqrt{n}}{2} (\hat{\lambda} - 1)^2 \frac{\partial^2 V_n(t; a, \nu)}{\partial \nu^2} \Big|_{\nu = \lambda_1}$$

and λ_1 lies between 1 and $\hat{\lambda}$. We need to establish that $\sup_{t \in [0,1]} |R_n(t)|$ is an $o_P(1)$ sequence. It can be shown that

$$\sqrt{n}(\hat{\lambda}-1) \xrightarrow{n \to \infty} \mathcal{N}(0,2),$$
 (10)

and that $\hat{\lambda}$ is a consistent estimator of 1 (see [3]). From (10) and from Slutsky's theorem we get $\sqrt{n}(\hat{\lambda}-1)^2 \xrightarrow{P} 0$. We have that

$$\begin{aligned} \frac{\partial^2 \Psi(X,Y;t,a,\nu)}{\partial \nu^2}\Big|_{\nu=\lambda_1} &= \frac{t^a (-\log(t))^{5/2}}{16\lambda_1^4} (-8Xt^{\frac{X}{\lambda_1}} (2\lambda_1 + X\log(t)) - 8Yt^{\frac{Y}{\lambda_1}} (2\lambda_1 + Y\log(t)) \\ &+ (X+Y)t^{\frac{X+Y}{4\lambda_1}} (8\lambda_1 + \log(t)(X+Y))). \end{aligned}$$

If we denote with

$$g(t;X) = \frac{8Xt^{\frac{X}{\lambda_1}}(2\lambda_1 + X\log(t))}{16\lambda_1^4},$$

then the following holds

$$g(t;X)| \leq \frac{1}{\lambda_1} \Big(\Big| \frac{Xt^{\frac{X}{\lambda_1}}}{\lambda_1^2} \Big| + \Big| \frac{X^2t^{\frac{X}{\lambda_1}}\log(t)}{2\lambda_1^3} \Big| \Big).$$

We have that

$$\left|\frac{Xt^{\frac{X}{\lambda_1}}}{\lambda_1^2}\right| = \frac{Xt^{\frac{X}{\lambda_1}}}{\lambda_1^2} = \frac{U^2 e^{U\log(t)}}{X}, \ U = \frac{X}{\lambda_1}$$

and noting that $\log(t) < 0$, we obtain that for every fixed t function $h(u) = u^2 e^{u \log(t)}$ attains its maximum at the point $u = -\frac{2}{\log(t)}$. Therefore,

$$\frac{Xt^{\frac{\lambda_1}{\lambda_1}}}{\lambda_1^2} \le \frac{4e^{-2}(-\log(t))^{-2}}{X}.$$
(11)

Similarly, we get that

$$\left|\frac{X^2 t^{\frac{X}{\lambda_1}} \log(t)}{2\lambda_1^3}\right| \le \frac{27e^{-3}(-\log(t))^{-2}}{2X}.$$
(12)

Since $\lambda_1 \ge \min(1, \hat{\lambda})$, we get that $\frac{1}{\lambda_1} \le \max\left(1, \frac{1}{\hat{\lambda}}\right)$. From (11) and (12) we conclude that

$$|g(t;X)| \le \frac{(4e+13.5)\max\left(1,\frac{1}{\tilde{\lambda}}\right)}{Xe^3(-\log(t))^2}.$$
(13)

Function $\omega(t;a) = t^a(-\log(t))^{\frac{1}{2}}$ attains its maximum as a continuous function of t on a compact set. Applying (13) three times, we get

$$\begin{split} \left| \frac{\partial^2 \Psi(X,Y;t,a,\nu)}{\partial \nu^2} \right|_{\nu=\lambda_1} \left| = t^a (-\log(t))^{5/2} |2g\left(t;\frac{X+Y}{4}\right) - g(t;X) - g(t;Y))| \\ &\leq t^a (-\log(t))^{5/2} (2|g\left(t;\frac{X+Y}{4}\right)| + |g(t;X)| + |g(t;Y))| \leq \max_{t\in[0,1]} |t^a (-\log(t))^{\frac{1}{2}}|(4e+13.5)e^{-3} \times \max\left(1,\frac{1}{\hat{\lambda}}\right) \left(\frac{8}{X+Y} + \frac{1}{X} + \frac{1}{Y}\right) = C(a) \max\left(1,\frac{1}{\hat{\lambda}}\right) \left(\frac{8}{X+Y} + \frac{1}{X} + \frac{1}{Y}\right). \end{split}$$
that that $C(a), \frac{1}{a} \sum_{i} \frac{1}{Y_i}$, and $\frac{1}{a^2} \sum_{i} \frac{1}{Y_i+Y_i}$ are $O_P(1)$ sequences.

Note that that C(a), $\frac{1}{n} \sum_{i} \frac{1}{X_i}$, and $\frac{1}{n^2} \sum_{i,j} \frac{1}{X_i + X_j}$ are $O_P(1)$ sequences.

Using the continuous mapping theorem, we get $\max\left(1, \frac{1}{\lambda}\right) \xrightarrow{P} 1$. Using the law of large numbers for V- statistics, the triangle inequality and Slutsky's theorem, we have that

$$\sup_{t \in [0,1]} \left| \frac{\partial^2 V_n(t;a,\nu)}{\partial \nu^2} \right| \le \sup_{t \in [0,1]} \frac{1}{n^2} \sum_{i_1,i_2}^n \left| \frac{\partial^2 \Psi(X_{i_1}, X_{i_2}; t, a, \nu)}{\partial \nu^2} \right| \le C(a) \max\left(1, \frac{1}{\hat{\lambda}}\right) \frac{1}{n^2} \sum_{i_1,i_2}^n \left(\frac{8}{X_{i_1} + X_{i_2}} + \frac{1}{X_{i_1}} + \frac{1}{X_{i_2}}\right) \xrightarrow{P} C(a) E\left(\frac{8}{X+Y} + \frac{1}{X} + \frac{1}{Y}\right) = 4C(a).$$

We have established that $\sup_{t \in [0,1]} \left| \frac{\partial^2 V_n(t;a,\nu)}{\partial \nu^2} \right|$ is an $O_P(1)$ sequence. Slutsky's theorem, along with the fact that $\sqrt{n}(\hat{\lambda}-1)^2$ is an $o_P(1)$ sequence establishes that $\sup_{t \in [0,1]} |R_n(t)|$ is an $o_P(1)$ sequence.

Using the law of large numbers for V- statistics once more, we have that

$$\frac{\partial V_n(t;a,\nu)}{\partial \nu}\Big|_{\nu=1} \xrightarrow{P} t^a (\log t)^{\frac{3}{2}} E\left(\frac{1}{2}X_1 t^{X_1} + \frac{1}{2}X_2 t^{X_2} - \frac{1}{4}(X_1 + X_2)t^{\frac{1}{4}(X_1 + X_2)}\right) = 0$$

The statement of the convergence $\sup_{t \in [0,1]} \left| \frac{\partial V_n(t;a,\nu)}{\partial \nu} \right|_{\nu=1} \right| \xrightarrow{P} 0$, is formalized in the following lemma.

Lemma 1. The limit in probability under H_0 of $\sup_{t \in [0,1]} \left| \frac{\partial V_n(t;a,\nu)}{\partial \nu} \right|_{\nu=1}$, as $n \to \infty$, equals 0.

Proof. Assume $a \ge 1$. Let's focus on the derivative of the kernel. The following holds:

$$\frac{\partial\Psi(X_1, X_2; t, a, \nu)}{\partial\nu}\Big|_{\nu=1} = \frac{t^a (-\log(t))^{5/2} \left(2X_1 t^{X_1} + 2X_2 t^{X_2} - (X_1 + X_2) t^{\frac{X_1 + X_2}{4}}\right)}{4}$$

Denote with

$$f_n(t) = \frac{1}{n^2} \sum_{i,j} \frac{\partial \Psi(X_1, X_2; t, a, \nu)}{\partial \nu} \Big|_{\nu=1} = \frac{1}{n^2} \sum_{i,j} \frac{t^a (-\log(t))^{5/2} \left(2X_1 t^{X_1} + 2X_2 t^{X_2} - (X_1 + X_2) t^{\frac{X_1 + X_2}{4}}\right)}{4}.$$

Function f_n is continuous (as a function of t) and has a continuous derivative. Moreover, for every $t \in [0, 1]$ we have that $Ef_n(t) = 0$. Denote with $S_{ND} = \{t : f'_n(t) \ge 0\}$, $S_{NI} = \{t : f'_n(t) \le 0\}$ the sets on which f_n is non-decreasing and non-increasing respectively. The continuity of $f'_n(t)$ ensures that both sets are closed subsets of the compact set [0, 1]. Therefore, S_{ND} and S_{NI} are compact. Note that $S_{ND} \cup S_{NI} = [0, 1]$.

From the subadditivity of the supremum, we have that

$$\sup_{t \in [0,1]} |f_n(t)| \le \sup_{t \in S_{ND}} |f_n(t)| + \sup_{t \in S_{NI}} |f_n(t)|$$

From the law of large numbers and continuity of the modulus, we have that for every $t \in [0, 1]$:

$$|f_n(t) - Ef_n(t)| = |f_n(t)| \xrightarrow{P} 0$$

Function $f_n(t)$ is non-decreasing for every $t \in S_{ND}$. By applying Lemma 1 from [29], we obtain

$$\sup_{t \in S_{ND}} |f_n(t)| \xrightarrow{P} 0$$

Similarly, function $f_n(t)$ is non-increasing for every $t \in S_{NI}$. By applying Lemma 1 from [29], we obtain

$$\sup_{t \in S_{ND}} |f_n(t)| \xrightarrow{P} 0.$$

Therefore, $\sup_{t \in [0,1]} \left| \frac{\partial V_n(t;a,\nu)}{\partial \nu} \right|_{\nu=1} \right| \leq \sup_{t \in S_{ND}} |f_n(t)| + \sup_{t \in S_{NI}} |f_n(t)| \to 0.$ The result then follows. \Box

Using Lemma 1 and using Slutsky's theorem, we conclude that under $H_0 \sqrt{n}V_n(t; a, \hat{\lambda})$ and $\sqrt{n}V_n(t; a, 1)$ are asymptotically equally distributed.

The distribution of $\sqrt{n}V_n(t; a, 1)$ can be obtained from Hoeffding theorem for non-degenerate U- (V-) statistics (see, e.g., [15]).

The first projection of the kernel $\Psi(\cdot; t, a)$ is given by

$$\psi(x;t,a) = E(\Psi(X_1, X_2; t, a | X_1 = x)) = \frac{1}{2} t^a (-\log(t))^{3/2} \Big(-2t^{\frac{x}{4}} e^{-\frac{\sqrt{-\log(t)}}{\sqrt{2}}} + t^x + e^{-\sqrt{2}\sqrt{-\log(t)}} \Big).$$

This function is obviously non-constant. In addition, it can be shown that $E\psi(X;t,a)^2 < \infty$ for every $t \in [0,1]$. Hence $V_n(t;a,1)$ is non-degenerate. Therefore, from the Hoeffding theorem and the multivariate central limit theorem, it follows that the finite-dimensional asymptotic distributions of $\sqrt{n}V_n(t;a,1)$ are normal. Hence, it suffices to show that the sequence $\{\sqrt{n}V_n(t;a,1)\}$ is tight. For the sake of brevity, we will denote $V_n(t;a,1)$ with $V_n(t;a)$ and we will drop the argument for λ in the following text whenever it is equal to 1. The tightness then follows from [7, Theorem 12.3].

Let us denote

$$V_n(t;a) = \frac{1}{n^2} \sum_{i,j} \Psi(X_i, X_j; t, a),$$

where Ψ denotes the symmetric kernel of the V-statistic.

To show tightness, we observe that

$$V_n(t+u;a) - V_n(t;a) = \frac{1}{n^2} \sum_{i,j} \left(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a) \right).$$

Therefore

$$E\Big(\sqrt{n}V_n(t+u;a) - \sqrt{n}V_n(t;a)\Big)^2 = E\Big(\frac{1}{n^3}\sum_{i,j,k,l}(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a))(\Psi(X_k, X_l; t+u, a) - \Psi(X_k, X_l; t, a))\Big).$$

Several different cases occur:

• If the indices i, j, k, l are different, the independence of the random variables X_i, X_k, X_j, X_l and the characterization give us

$$E\Big(\frac{1}{n^3}\sum_{i\neq j\neq k\neq l}(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a))(\Psi(X_k, X_l; t+u, a) - \Psi(X_k, X_l; t, a))\Big) = 0,$$

and we have $4!\binom{n}{4}$ such cases.

- If three out of four indices are different, then two cases can occur:
 - 1. If i = j, then the independence and characterization give us

$$E\Big(\frac{1}{n^3}\sum_{i\neq k\neq l}(\Psi(X_i, X_i; t+u, a) - \Psi(X_i, X_i; t, a))(\Psi(X_k, X_l; t+u, a) - \Psi(X_k, X_l; t, a))\Big) = 0,$$

and we have $8n\binom{n-1}{2} = 4n(n-1)(n-2)$ such cases. 2. If without loss of generality i = k, then we have that

$$E\left(\frac{1}{n^3}\sum_{i\neq j\neq l}(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a)) \quad (\Psi(X_i, X_l; t+u, a) - \Psi(X_i, X_l; t, a))\right) \neq 0.$$

Since we have $8n\binom{n-1}{2} = 4n(n-1)(n-2)$ such cases, the sum above reduces to

$$\frac{4n(n-1)(n-2)}{n^3} E\left(\left(\Psi(X_1, X_2; t+u, a) - \Psi(X_1, X_2; t, a)\right)(\Psi(X_1, X_3; t+u, a) - \Psi(X_1, X_3; t, a))\right)$$

= $\frac{4(n-1)(n-2)}{n^2} E(\psi(X_1; t+u, a) - \psi(X_1; t, a))^2,$

where $\psi(X; t, a)$ denotes the first projection of the kernel Ψ .

• If two out of four indices are different, then we have three different cases.

1. If i = k and j = l, we have that

$$E\Big(\frac{1}{n^3}\sum_{i\neq j}(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a))^2\Big),$$

and since it reduces to

$$\begin{aligned} &\frac{4}{n^3} \binom{n}{2} E(\Psi(X_1, X_2; t+u, a) - \Psi(X_1, X_2; t, a))^2 = \\ &\frac{2(n-1)}{n^2} E(\Psi(X_1, X_2; t+u, a) - \Psi(X_1, X_2; t, a))^2 \xrightarrow{n \to \infty} 0, \end{aligned}$$

we conclude that this part is asymptotically negligible.

2. If i = j and k = l, we have that

$$E\left(\frac{1}{n^3}\sum_{i\neq j}(\Psi(X_i, X_i; t+u, a) - \Psi(X_i, X_i; t, a))(\Psi(X_j, X_j; t+u, a) - \Psi(X_j, X_j; t, a))\right)$$

= $\frac{2n(n-1)}{n^3}E(\Psi(X_1, X_1; t+u, a) - \Psi(X_1, X_1; t, a))^2,$

and since

$$E(\Psi(X_1, X_1; t+u, a) - \Psi(X_1, X_1; t, a))^2 < \infty,$$

the asymptotic negligence follows.

3. If $i = k = l \neq j$, then we have that the independence and characterization give us

$$E\left(\frac{1}{n^3}\sum_{i\neq j}(\Psi(X_i, X_i; t+u, a) - \Psi(X_i, X_i; t, a))(\Psi(X_i, X_j; t+u, a) - \Psi(X_i, X_j; t, a))\right) = 0.$$

and we have 2n(n-1) such cases.

• The only remaining possibility is that all indices coincide. We have that

$$E\left(\frac{1}{n^3}\sum_{i}(\Psi(X_i, X_i; t+u, a) - \Psi(X_i, X_i; t, a))\right)^2 = E\left(\frac{1}{n^2}(\Psi(X_i, X_i; t+u, a) - \Psi(X_i, X_i; t, a))^2\right)^2$$

and since

$$E(\Psi(X_1, X_1; t+u, a) - \Psi(X_1, X_1; t, a))^2 < \infty,$$

the asymptotic negligence follows.

We have that as $n \to \infty$

$$H(u) = E\left(\sqrt{n}V_n(t+u;a) - \sqrt{n}V_n(t;a)\right)^2 \to 4E(\psi(X_1;t+u,a) - \psi(X_1;t,a))^2.$$

Exploiting the mean-value theorem gives us

$$E(\psi(X_1; t+u, a) - \psi(X_1; t, a))^2 \le E\left(\frac{d\psi(X_1; t_1, a)}{dt}\right)^2 u^2,$$

for $t_1 \in [t, t+u]$. We have that $E\left(\frac{d\psi(X_1;t_1,a)}{dt}\right)^2$ is continuous function for $t \in [0,1]$ and $a \ge 1$. Taking into account that the set [0,1] is compact, we have that the following quantity

$$C = \max_{t \in [0,1]} E\left(\frac{d\psi(X_1; t, a)}{dt}\right)^2$$

exists and is finite. Therefore, tightness follows from $H(u) \leq Cu^2$.

We have established that $\sqrt{n}J_{n,a} \xrightarrow{D} \sup_{t \in [0,1]} |\xi(t)|$, where $\{\xi(t)\}$ is the centred Gaussian process, whose covariance function can be obtained from the following:

$$K(s,t) = \frac{s^{a}t^{a}}{4}(-\log(s))^{3/2}(-\log(t))^{3/2}E(\Psi(X,Y;t,a)\Psi(X,Z;s,a)) = \frac{s^{a}t^{a}}{4}(-\log(s))^{3/2}(-\log(t))^{3/2} \times (14)$$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\left(-2s^{\frac{x+z}{4}} + s^z + s^x\right) \left(-2t^{\frac{x+y}{4}} + t^y + t^x\right) e^{\frac{1}{2}\left(-\frac{1}{x} - \frac{y+z}{yz}\right)}}{8\sqrt{2}\pi^{3/2} x^{3/2} y^{3/2} z^{3/2}} dx dy dz.$$

The result of the computation above is stated in the Theorem 1.

Proof of Theorem 2. Analogously as before, the statistic (4) can be represented as

$$R_{n,a} = V_n^R(a,\hat{\lambda}),$$

where, for each t and a,

$$V_n^R(a,\hat{\lambda}) = \frac{1}{n^2} \sum_{i_1,i_2}^n Z(X_{i_1}, X_{i_2}; a, \hat{\lambda}),$$

is a V-statistic with the estimated parameter λ , with symmetric kernel $Z(\cdot; a, \hat{\lambda})$. Applying Taylor expansion as in (9) and using (10), we conclude that the normality of $\sqrt{n}V_n^R(a, \hat{\lambda})$ can be obtained from the asymptotic normality of $\sqrt{n}V_n^R(a)$ applying Hoeffding theorem for non-degenerate U- (V-) statistics (see, e.g., [15]).

The first projection of the kernel $Z(\cdot; a) := Z(\cdot; a, 1)$ is given by:

$$\begin{aligned} \zeta(x;a) &= E(Z(X_1, X_2;a) | X_1 = x) = -\frac{\sqrt{\pi} (3a(a+2)+1)e^{\frac{1}{2a}} \operatorname{erf}\left(\frac{1}{\sqrt{2a}}\right) - \sqrt{2}\sqrt{a}(5a+1)}{16a^{9/2}} \\ &+ \frac{\left(8\pi e^{\frac{1}{8a+2x}} \left(48a^2 + 24a(x+1) + 3x(x+2) + 1\right)\operatorname{erf}\left(\frac{1}{\sqrt{2(4a+x)}}\right)}{(4a+x)^{9/2}} - \frac{8\sqrt{2\pi}(20a+5x+1)\right)}{(4a+x)^4} - \frac{3\sqrt{\pi}}{8(a+x)^{5/2}}. \end{aligned}$$
(15)

This function is obviously non-constant. In addition, it can be shown that $E\zeta(X;a)^2 < \infty$ for every $t \in [0,1]$. Hence $V_n^R(a,1)$ is non-degenerate. Analogously as before, utilizing the Hoeffding theorem and the multivariate central limit theorem, we obtain that the limiting distribution of $\sqrt{n}V_n^R(a, \hat{\lambda})$ is normal $\mathcal{N}(0, \sigma_R^2(a))$, where $\sigma_R^2(a) = 4E\zeta(X;a)^2$.

Proof of Theorem 3. It can be easily shown that the minimum of $K(\theta; \lambda)$, as a function of λ , is attained for

$$\lambda_0 = \left(\int\limits_{\mathbb{R}^+} \frac{g(x;\theta)}{x} dx\right)^{-1}.$$

Therefore

$$K(\theta) = \int_{\mathbb{R}^+} \left(\log\left(\frac{g(x;\theta)}{f(x;\lambda_0)}\right) g(x;\theta) \right) dx.$$

$$K(\theta) = \int_{\mathbb{R}^+} \log(g(x;\theta))g(x;\theta)dx + \log\sqrt{2\pi} + \frac{1}{2}\log\Big(\int_{\mathbb{R}^+} \frac{g(x;\theta)}{x}dx\Big) + \frac{1}{2} + \frac{3}{2}\int_{\mathbb{R}^+} \log xg(x;\theta)dx.$$

Under certain regularity conditions [26], the following holds:

$$K'(\theta) = \int_{\mathbb{R}^+} \log(g(x;\theta))g'(x;\theta)dx + \frac{\int_{\mathbb{R}^+} \frac{g'(x;\theta)}{x}dx}{2\int_{\mathbb{R}^+} \frac{g(x;\theta)}{x}dx} + \frac{3}{2}\int_{\mathbb{R}^+} \log xg'(x;\theta)dx.$$

Noting that

$$g(x;0) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda}{2x}} x^{-\frac{3}{2}},$$

where $\lambda > 0$ is a parameter, it can be shown that K'(0) = 0.

The second derivative of $K(\theta)$ is equal to

$$\begin{split} K''(\theta) &= \int\limits_{\mathbb{R}^+} \frac{(g'(x;\theta))^2}{g(x;\theta)} dx + \int\limits_{\mathbb{R}^+} \log(g(x;\theta))g''(x;\theta) dx \\ &+ \frac{\left(\int\limits_{\mathbb{R}^+} \frac{g''(x;\theta)}{x} dx\right) \left(\int\limits_{\mathbb{R}^+} \frac{g(x;\theta)}{x} dx\right) - \left(\int\limits_{\mathbb{R}^+} \frac{g'(x;\theta)}{x} dx\right)^2}{2\left(\int\limits_{\mathbb{R}^+} \frac{g(x;\theta)}{x} dx\right)^2} + \frac{3}{2} \int\limits_{\mathbb{R}^+} \log(x)g''(x;\theta) dx. \end{split}$$

The straightforward computation leads us to the following:

$$K''(0) = \sqrt{\frac{2\pi}{\lambda}} \int_{\mathbb{R}^+} (g'(x;\theta))^2 e^{-\frac{\lambda}{2x}} x^{-\frac{3}{2}} dx - \frac{\lambda}{2} \Big(\int_{\mathbb{R}^+} \frac{g'(x;\theta)}{x} dx \Big)^2,$$

and the conclusion follows from the Maclaurin expansion of $K(\theta)$.

Proof of Theorem 5. We give just a broad outline of the proof. The tail behaviour of $\sup_{t \in [0,1]} |\xi(t)|$ is equal to the inverse of supremum of its covariance function [19]. Having $\hat{\lambda} \xrightarrow{P} \lambda(\theta)$, the law of large numbers for V-statistics for estimated parameters [13] gives us that $V_n(t; a, \hat{\lambda})$ converges to $A_J(\theta; t) = E_{\theta}(\Psi(X_1, X_2; t, a, \lambda(\theta)))$. By expanding into the Maclaurin series, we get

$$A_J(\theta;t) = A_J(0;t) + \theta A'_J(\theta;t) + \frac{\theta^2}{2} A''_J(0;t) + o(\theta^2).$$

It can be shown that $A_J(0;t) = b'_J(0) = 0$. The direct calculation yields

$$A_J''(0;t) = 2 \int_{\mathbb{R}^+} \psi(x;t,a) g_\theta'(x;0) dx.$$

Using the same arguments as in the proof of Theorem 1, it can be shown that the limit in probability of $J_{n,a}$ under the alternative equals to $\sup_{t \in [0,1]} |A_J(\theta;t)| = b_J(\theta)$, which finishes the proof.

Proof of Theorem 6. The theorem can be shown analogously as in [23, Lemma 2.1]. Hence we omit it here. \Box

Appendix B – The generalization of the Bhati–Kattumanil statistic

One possible way to generalize the test statistic T_n is to opt for the difference between U-empirical distribution functions of $\omega_1(X_1, X_2) = \frac{aX_1 + bX_2}{\left(\sqrt{a} + \sqrt{b}\right)^2}$ and $\omega_2(X_1) = X_1$, given by

$$G_n(t) = \frac{1}{n(n-1)} \sum_{i < j} \mathbf{I} \left\{ \frac{aX_i + bX_j}{(\sqrt{a} + \sqrt{b})^2} \le t \right\} \text{ and}$$
$$F_n(t) = \frac{1}{n} \sum_i \mathbf{I} \left\{ X_i \le t \right\}$$

respectively. This leads us to the statistic

$$\bar{I}_{n}^{[a,b]} = \int_{\mathbb{R}^{+}} (G_{n}(t) - F_{n}(t)) dF_{n}(t).$$
(16)

It is easy to show that the statistic is scale free under the null hypothesis. This follows from the fact that if X has the Lévy distribution with the scale parameter λ , then $\frac{X}{\lambda}$ has the standard Lévy distribution. Therefore, while deriving asymptotic null properties we may assume that the sample comes from the standard Lévy distribution. Applying the same arguments as in [6], we come to the following statement about the limiting distribution of $\overline{I}_n^{[a,b]}$ under H_0 .

Theorem 7. Let $X_1, X_2, ..., X_n$ be an i.i.d. sample. Under \mathcal{H}_0 the limiting distribution of $\overline{I}_n^{[a,b]}$ is centred Gaussian, *i.e.* it holds

$$\sqrt{n}\bar{I}_n^{[a,b]} \xrightarrow{D} \mathcal{N}(0,\sigma_0^2(a,b))$$

where

$$\sigma_0^2(a,b) = Var\Big(2 - P\Big(\frac{aX_1 + bX_2}{(\sqrt{a} + \sqrt{b})^2} \ge X_3 | X_1\Big) - P\Big(\frac{aX_2 + bX_1}{(\sqrt{a} + \sqrt{b})^2} \ge X_3 | X_1\Big) + P\Big(X_2 \le X_1 | X_1\Big)\Big).$$
(17)

The values of $\sigma_0^2(a, b)$ cannot be calculated analytically. However, it is possible to calculate them numerically. Other values of $\sigma_0^2(a, b)$ are presented in Table 5. Therefore, instead of using jackknife approach, one can also test using standardized statistic

$$\widetilde{I}_n^{[a,b]} = \sqrt{n} \frac{\overline{I}_n^{[a,b]}}{\sigma_0(a,b)},$$

or calculate *p*-values based on $\bar{I}_n^{[a,b]}$ using Monte Carlo approach. Both mentioned approaches are much simpler than original proposed in [6].

Table 5: Some values of $\sigma_0^2(a, b)$

a	b	σ_0^2	а	b	σ_0^2	а	b	σ_0^2	a	b	σ_0^2
1	2	0.022621	3	10	0.0209729	2	5	0.02199	5	10	0.022621
1	3	0.0213695	4	5	0.0234113	2	6	0.0213695	6	7	0.0234603
1	4	0.0202296	4	6	0.0231973	2	7	0.0207807	6	8	0.0233495
1	5	0.0192384	4	7	0.0229236	2	8	0.0202296	6	9	0.0231973
1	6	0.0183778	4	8	0.022621	2	9	0.0197162	6	10	0.0230191
1	7	0.0176251	4	9	0.0223067	2	10	0.0192384	7	8	0.0234715
1	8	0.0169606	4	10	0.02199	3	4	0.0233495	7	9	0.0233862
1	9	0.0163687	5	6	0.0234425	3	5	0.0230191	7	10	0.0232665
1	10	0.0158373	5	7	0.0232926	3	6	0.022621	8	9	0.023479
2	3	0.0231973	5	8	0.0230928	3	7	0.022201	8	10	0.0234113
2	4	0.022621	5	9	0.0228647	3	8	0.0217804	9	10	0.0234842
3	9	0.0213695	10	10	0.0235051						

Analogously to Theorem 4, we formulate the result in the general case.

Theorem 8. For an alternative $g(x; \theta)$ from \mathcal{G} , the Bahadur exact slope of the statistic $I_n^{[a,b]}$ is

$$c_I(\theta) = \frac{1}{\sigma_0^2(a,b)} \Big(\int_{\mathbb{R}^+} \varphi(x) g'_{\theta}(x;0) dx \Big)^2 \cdot \theta^2 + o(\theta^2), \theta \to 0,$$

where $\varphi(x)$ is the first projection of the symmetric kernel $\Phi(\cdot)$ of V-statistic that is asymptotically equivalent to $I_n^{[a,b]}$, namely

$$\varphi(x) = \left(2 - P\left(\frac{aX_1 + bX_2}{(\sqrt{a} + \sqrt{b})^2} \ge X_3 | X_1\right) - P\left(\frac{aX_2 + bX_1}{(\sqrt{a} + \sqrt{b})^2} \ge X_3 | X_1\right) + P\left(X_2 \le X_1 | X_1\right)\right).$$

There is no significant difference between the statistic $I^{[1,1]}$ and $I^{[a,b]}$ for different values of a and b with regard to the empirical powers against all of the alternatives mentioned in this paper and the local approximate Bahadur relative efficiencies, as can be seen in Table 6.

Table 6: Local approximate Bahadur relative efficiencies of $I^{[a,b]}$ with respect to LR test

	$g_{1}^{[10]}$	g_2	$g_{3}^{[3]}$	g_4	g_5
$I^{[1,1]}$	0.59	0.54	0.73	0.53	0.41
$I^{[2,3]}$	0.59	0.54	0.73	0.53	0.41
$I^{[5,9]}$	0.58	0.54	0.73	0.53	0.41
$I^{[9,6]}$	0.59	0.54	0.73	0.53	0.41
$I^{[10,4]}$	0.57	0.53	0.72	0.52	0.40

Appendix C – Real data

In this section, the data used in Section 6 is given alongside with appropriate histograms. The theoretical Lévy densities are drawn using the maximum likelihood estimate of the scale parameter λ .



Figure 3: Histogram of the data from Table 7 and the appropriate Lévy density. The purple line represents the Lévy density with the scale parameter estimated by MLE ($\hat{\lambda} = 11.82935$).

Year	Rainfall	Year	Rainfal
1981	29.3	1997	14.3
1982	23.8	1998	16.4
1983	18.5	1999	13.7
1984	19	2000	18.4
1985	23.2	2001	7.3
1986	15.5	2002	15.7
1987	13.2	2003	7.6
1988	10.4	2004	25.7
1989	15.4	2005	28.1
1990	16	2006	17.7
1991	14.3	2007	1.7
1992	16	2008	18.4
1993	18.2	2009	12
1994	25	2010	7.5
1995	31.3	2011	6.8
1996	22.9		
-			

Table 7: Weighted average of rainfall (in mm) data for India for the month of January

Table 8: Well yields (in gal/min/ft) based on Hillside location

1			-			
	0.220	1.330	0.750	0.180	0.010	0.160
	0.280	0.870	0.020	0.100	0.030	0.050
	0.860	5.000	0.040	4.000	0.370	0.380
	0.110	0.100	0.020	0.010	0.050	0.170
	0.460	0.160	1.330	0.140	2.860	0.130
	7.500	4.500	0.030	0.003	0.050	0.020
	0.040	0.750	0.520	5.000	0.350	



Figure 4: Histogram of the inverse of the data from Table 8 and the appropriate Lévy density. The red line represents the Lévy density with the scale parameter estimated by MLE ($\hat{\lambda} = 1.052551$).

Appendix D – Median-based estimator

We conduct the power study as in Section 5 when the median-based estimator is employed. Results are presented in Table 10. From Table 10, it can be seen that JEL and AJEL approaches, proposed in [6], are less powerful than classical, whenever the testing is utilized via the original version of $|I^{[1,1]}|$. It can be concluded that novel tests are comparable with the tests N_1^a and N_1^b proposed in [32]. It is also notable that the new tests are comparable with EDFbased tests. In many cases, the new tests show better performance than the EDF-based tests. From Tables 11, 12, 13, and 14 seems that the estimation procedure doesn't significantly influence the distribution under the null hypothesis for larger sample sizes for the novel tests, which is in concordance with previously obtained theoretical results. The significant difference in test powers in Table 3 and 10 could be attributed to the difference in the behaviour of the estimates under the alternative distributions.

Note that results analogous to Theorem 1 and 2 could be established similarly to the MLE case. The consistency of $\hat{\lambda}_{MBE}$ will follow from [20].

The novel tests using the MBE can be applied to the real data examples from Section 6. Results are presented in Table 9.

Tab	le 9: <i>p</i> -val	lues of no	vel tests -	MB estim	ate
	$R_{0.2}$	$R_{0.5}$	R_1	R_2	R_5
Rainfall Hillside	0.3559 0.012	0.0314 0.0058	0 0.042	0 0.4087	0 0.6572
	J_1	J_2	J_5	J_{10}	
Rainfall Hillside	0.026 0.014	0 0.1364	0 0.7029	0 0.3086	

From Figure 3 presented in Appendix C, it can be deduced that the empirical density of the Rainfall data is, among the distributions studied in the simulation study, closest to LG(7, 2). Since $R_{0,2}$ is the least powerful test against this alternative and all of the other tests report *p*-values smaller than 0.05, we can conclude that the Lévy distribution is not a justified choice for the Rainfall data.

Analogously to the MLE case, $R_{0,2}$ is quite powerful against LL(1, 2). Therefore, we cannot conclude that the Lévy distribution is the appropriate model for the Hillside data.

	N_1^b	0.05	0.06	0.05	0.44	0.36	0.57	0.61	0.78	0.86	0.03	0.05	0.03	0.05	0.20	0.09	0.16	0.05	0.04	0.05	0.72	0.76	0.94	0.96	1	1	0.05	0.04	0.04	0.09	0.46	0.19	
	N_1^a	0.05	0.05	0.05	0.19	0.30	0.58	0.73	0.73	0.87	0.21	0.07	0.07	0.25	0.16	0.10	0.11	0.05	0.05	0.05	0.27	0.74	0.98	1	1	1	0.65	0.18	0.16	0.72	0.37	0.21	
	AD	0.05	0.05	0.06	1	0.40	0.74	0.83	1	1	0.95	0.98	0.44	0.89	0.05	0.33	0.19	0.06	0.04	0.05	1	0.99	1	1	1	1	1	1	0.72	1	0.29	0.57	
	CVM	0.04	0.04	0.05	0.86	0.78	0.95	0.96	1	1	0.40	0.49	0.10	0.34	0.21	0.08	0.44	0.05	0.05	0.05	1	1	1	1	1	1	0.80	0.88	0.22	0.73	0.48	0.18	000
	KS	0.05	0.05	0.05	0.83	0.60	0.94	0.97	1	1	0.38	0.55	0.11	0.29	0.15	0.06	0.30	0.05	0.05	0.05	0.99	0.98	1	1	1	1	0.79	0.91	0.20	0.67	0.32	0.14	0.62
	AJEL	0.04	0.04	0.04	0.14	0.21	0.36	0.35	0.72	0.73	0.05	0.05	0.04	0.05	0.07	0.05	0.09	0.04	0.04	0.04	0.26	0.48	0.71	0.69	0.98	0.99	0.06	0.06	0.04	0.06	0.14	0.06	0.10
stimate	JEL	0.05	0.06	0.05	0.18	0.25	0.41	0.40	0.76	0.77	0.06	0.07	0.06	0.07	0.09	0.06	0.11	0.05	0.05	0.05	0.29	0.52	0.74	0.72	0.99	0.99	0.07	0.07	0.05	0.07	0.16	0.06	100
- MB e	R_5 .	0.05	0.05	0.05	0.87	66.0	-	-	1	1	0.12	0.22	0.06	0.22	0.51	0.18	0.69	0.05	0.05	0.06	1	T	1	1	1	1	0.16	0.40	0.06	0.33	0.82	0.31	100
powers	R_2	0.05	0.05	0.05	0.97	0.94	0.98	0.95	1	1	0.21	0.63	0.06	0.11	0.38	0.07	0.67	0.05	0.05	0.05	1	1	1	1	1	1	0.38	0.92	0.10	0.15	0.67	0.08	0.07
pirical ₁	R_1	0.04	0.05	0.05	0.99	0.70	0.74	0.55	1	1	0.71	0.92	0.23	0.48	0.18	0.08	0.51	0.05	0.05	0.05	1	0.97	0.96	0.82	1	1	0.93	1	0.45	0.76	0.39	0.10	0.88
n of em	$R_{0.5}$	0.05	0.05	0.05	1	0.10	0.13	0.10	0.76	0.72	0.94	0.98	0.49	0.85	0.02	0.29	0.07	0.05	0.05	0.05	1	0.56	0.39	0.18	1	0.98	1	1	0.77	0.98	0.07	0.46	0 55
mparisc	$R_{0.2}$	0.05	0.05	0.05	1	0	0.07	0.28	0	0	0.98	0.99	0.65	0.94	0.01	0.52	0	0.05	0.04	0.04	1	0	0.10	0.42	0	0	1	1	0.90	1	0	0.79	0
10: Co	J_{10}	0.05	0.05	0.05	0.77	0.99	1	1	1	1	0.27	0.15	0.09	0.40	0.50	0.24	0.65	0.05	0.05	0.05	0.98	1	1	1	1	1	0.45	0.25	0.11	0.67	0.80	0.40	0.01
Table	J_5	0.05	0.05	0.05	0.84	0.97	1	1	1	1	0.15	0.31	0.05	0.19	0.45	0.17	0.66	0.05	0.05	0.04	0.99	1	1	1	1	1	0.23	0.62	0.05	0.28	0.73	0.25	0 07
	J_2	0.05	0.05	0.05	0.95	0.83	0.94	0.91	1	1	0.50	0.80	0.12	0.32	0.28	0.06	0.56	0.05	0.05	0.05	1	0.99	1	1	1	1	0.85	0.98	0.26	0.64	0.51	0.07	0.88
	J_1	0.05	0.05	0.05	0.99	0.36	0.50	0.41	0.99	0.99	0.89	0.96	0.40	0.77	0.05	0.19	0.23	0.05	0.05	0.05	1	0.83	0.90	0.82	1	1	0.99	1	0.67	0.97	0.17	0.34	0.68
	$\overline{I}^{[1,1]}$	0.05	0.05	0.05	0	060	1	1	1	1	0.15	0.01	0.07	0.28	0.56	0.22	0.74	0.05	0.05	0.05	0	1	1	1	1	1	0.27	0	0.08	0.52	0.9	0.42	0 07
	u	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
	Distribution	Lévy(0, 0.5)	Lévy(0, 1)	Lévy(0, 2)	Burr(1.5, 0.5, 0.5)	LN(0,1)	$\chi^2(3)$	HN(0,1)	$\Gamma(3,2)$	W(2, 1)	$\Gamma(0.4,2)$	W(0.4,2)	LN(0,2)	Chen(2, 0.4)	LG(7, 2)	LL(1, 2)	FR(1, 1)	Lévy(0, 0.5)	Lévy(0, 1)	Lévy(0, 2)	Burr(1.5, 0.5, 0.5)	LN(0, 1)	$\chi^2(3)$	HN(0,1)	$\Gamma(3,2)$	W(2, 1)	$\Gamma(0.4,2)$	W(0.4,2)	LN(0,2)	Chen(2, 0.4)	LG(7, 2)	LL(1, 2)	FR(1 1)

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Appendix E – Critical values of the new tests

In this section, the empirical 95th percentiles of the distributions of $\sqrt{n}J_{n,a}$ and $|\sqrt{n}R_{n,a}|$, under H_0 , are presented. Different shape parameters λ of the null distribution are used. The values are computed using Monte Carlo simulations with N = 100000 repetitions. In Tables 13 and 14, in the column ∞ , 95th percentiles of the asymptotic half-normal $HN(0, \sigma_R^2(a))$ distribution are presented.

n	$\sqrt{n}J_{n,1}$ MLE	$\sqrt{n}J_{n,1}$ MED	$\sqrt{n}J_{n,2}$ MLE	$\sqrt{n}J_{n,2}$ MED	$\sqrt{n}J_{n,5}$ MLE	$\sqrt{n}J_{n,5}$ MED	$\sqrt{n}J_{n,10}$ MLE	$\sqrt{n}J_{n,10}$ MED
20	0.14827	0.14335	0.02585	0.02465	0.00212	0.00215	0.00029	0.00029
40	0.14112	0.13843	0.02511	0.02483	0.00208	0.00209	0.00028	0.00029
60	0.13798	0.13679	0.02482	0.02450	0.00211	0.00209	0.00029	0.00029
80	0.14025	0.14030	0.02565	0.02504	0.00211	0.00210	0.00029	0.00029
100	0.13997	0.13954	0.02477	0.02467	0.00215	0.00214	0.00028	0.00029
120	0.13885	0.13779	0.02482	0.02499	0.00209	0.00208	0.00029	0.00029
140	0.13847	0.13828	0.02467	0.02455	0.00206	0.00205	0.00028	0.00029
160	0.13808	0.13750	0.02518	0.02483	0.00210	0.00208	0.00029	0.00029
180	0.14067	0.13904	0.02469	0.02473	0.00208	0.00210	0.00029	0.00029
200	0.14079	0.13972	0.02519	0.02505	0.00207	0.00206	0.00028	0.00028
220	0.13827	0.13856	0.02465	0.02444	0.00207	0.00207	0.00029	0.00029
240	0.14066	0.13747	0.02515	0.02493	0.00208	0.00207	0.00029	0.00029
260	0.13857	0.13848	0.02484	0.02472	0.00211	0.00211	0.00028	0.00028
280	0.14032	0.14023	0.02509	0.02500	0.00208	0.00208	0.00029	0.00028
300	0.13732	0.13691	0.02509	0.02475	0.00210	0.00209	0.00029	0.00029
320	0.13790	0.13780	0.02470	0.02470	0.00208	0.00207	0.00028	0.00029
340	0.13694	0.13777	0.02470	0.02478	0.00208	0.00208	0.00028	0.00028
360	0.14034	0.13795	0.02491	0.02491	0.00208	0.00207	0.00028	0.00028
380	0.14178	0.14103	0.02515	0.02497	0.00210	0.00210	0.00028	0.00028
400	0.13703	0.13746	0.02496	0.02479	0.00209	0.00209	0.00029	0.00029
420	0.13731	0.13813	0.02505	0.02495	0.00209	0.00209	0.00029	0.00028
440	0.13866	0.13864	0.02472	0.02486	0.00204	0.00205	0.00029	0.00029
460	0.13859	0.13753	0.02500	0.02494	0.00210	0.00211	0.00028	0.00028
480	0.13863	0.13743	0.02471	0.02455	0.00207	0.00207	0.00029	0.00029
500	0.13778	0.13719	0.02471	0.02461	0.00211	0.00210	0.00029	0.00029

Table 11: Critical values of $\sqrt{n}J_{n,a}$ statistic, for N=100000 repetitions and $\lambda = 0.5$.

Table 12: Critical values of $\sqrt{n}J_{n,a}$ statistic, for N=100000 repetitions and $\lambda = 5$.

n	$\sqrt{nJ_{n,1}}$ MLE	$\sqrt{nJ_{n,1}}$ MED	$\sqrt{nJ_{n,2}}$ MLE	$\sqrt{nJ_{n,2}}$ MED	$\sqrt{nJ_{n,5}}$ MLE	$\sqrt{nJ_{n,5}}$ MED	$\sqrt{nJ_{n,10}}$ MLE	$\sqrt{nJ_{n,10}}$ MED
20	0.14721	0.14587	0.02576	0.02487	0.00211	0.00211	0.00028	0.00028
40	0.14223	0.13780	0.02557	0.02467	0.00213	0.00214	0.00028	0.00028
60	0.14306	0.14023	0.02499	0.02454	0.00208	0.00207	0.00028	0.00028
80	0.14053	0.13881	0.02519	0.02472	0.00209	0.00210	0.00029	0.00029
100	0.14162	0.13724	0.02508	0.02492	0.00208	0.00208	0.00029	0.00029
120	0.13977	0.13679	0.02506	0.02467	0.00207	0.00208	0.00029	0.00029
140	0.13999	0.13811	0.02505	0.02498	0.00208	0.00207	0.00028	0.00029
160	0.13895	0.13837	0.02482	0.02485	0.00208	0.00208	0.00029	0.00029
180	0.13903	0.13740	0.02518	0.02474	0.00208	0.00211	0.00029	0.00028
200	0.14000	0.13887	0.02497	0.02479	0.00208	0.00207	0.00029	0.00029
220	0.13798	0.13963	0.02483	0.02455	0.00211	0.00213	0.00029	0.00029
240	0.13681	0.13660	0.02480	0.02461	0.00207	0.00206	0.00029	0.00029
260	0.13797	0.13741	0.02503	0.02470	0.00208	0.00208	0.00029	0.00029
280	0.14082	0.13991	0.02479	0.02484	0.00206	0.00208	0.00029	0.00029
300	0.13751	0.13719	0.02484	0.02457	0.00206	0.00206	0.00029	0.00029
320	0.14183	0.13939	0.02544	0.02521	0.00208	0.00208	0.00028	0.00028
340	0.14109	0.13875	0.02514	0.02509	0.00209	0.00208	0.00028	0.00028
360	0.13746	0.13723	0.02497	0.02493	0.00209	0.00210	0.00029	0.00029
380	0.14061	0.13883	0.02480	0.02477	0.00207	0.00206	0.00028	0.00028
400	0.13714	0.13641	0.02415	0.02409	0.00212	0.00211	0.00029	0.00029
420	0.13798	0.13886	0.02485	0.02459	0.00209	0.00209	0.00029	0.00029
440	0.14116	0.14154	0.02471	0.02454	0.00210	0.00210	0.00029	0.00029
460	0.14085	0.14010	0.02464	0.02468	0.00209	0.00209	0.00028	0.00028
480	0.13887	0.13834	0.02513	0.02505	0.00208	0.00208	0.00029	0.00029
500	0.13690	0.13597	0.02492	0.02454	0.00208	0.00209	0.00029	0.00029

	$ \sqrt{n}R_{n,5} $ MED	0.00868	0.00865	0.00864	0.00862	0.00867	0.00861	0.00864	0.00861	0.00860	0.00859	0.00855	0.00860	0.00859	0.00857	0.00856	0.00860	0.00857	0.00859	0.00857	0.00859	0.00860	0.00861	0.00866	0.00863	0.00860	0.00860
$d \lambda = 0.5.$	$ \sqrt{n}R_{n,5} $ MLE	0.00862	0.00864	0.00865	0.00858	0.00863	0.00861	0.00855	0.00864	0.00864	0.00860	0.00860	0.00862	0.00861	0.00866	0.00860	0.00865	0.00863	0.00861	0.00864	0.00862	0.00857	0.00858	0.00858	0.00857	0.00859	0.00860
epetitions an	$ \sqrt{n}R_{n,2} $ MED	0.06797	0.06794	0.06749	0.06746	0.06803	0.06776	0.06776	0.06772	0.06763	0.06805	0.06723	0.06759	0.06742	0.06750	0.06751	0.06802	0.06781	0.06734	0.06775	0.06774	0.06755	0.06770	0.06781	0.06808	0.06784	0.06774
· N=100000 r	$ \sqrt{n}R_{n,2} $ MLE	0.07024	0.06871	0.06834	0.06806	0.06825	0.06798	0.06754	0.06825	0.06798	0.06762	0.06781	0.06786	0.06824	0.06804	0.06790	0.06826	0.06788	0.06779	0.06798	0.06769	0.06801	0.06769	0.06782	0.06743	0.06776	0.06774
statistic, for	$ \sqrt{n}R_{n,1} $ MED	0.27948	0.28013	0.28032	0.28035	0.28131	0.28059	0.28178	0.28091	0.28069	0.28204	0.28047	0.28071	0.28062	0.28093	0.28109	0.28214	0.28254	0.28022	0.28085	0.28159	0.28272	0.28204	0.28072	0.28232	0.28304	0.28191
: of $ \sqrt{n}R_{n,a} $	$ \sqrt{n}R_{n,1} $ MLE	0.29687	0.28856	0.28602	0.28599	0.28447	0.28244	0.28174	0.28465	0.28290	0.28083	0.28281	0.28289	0.28399	0.28354	0.28188	0.28433	0.28352	0.28188	0.28328	0.28272	0.28367	0.28278	0.28326	0.28094	0.28273	0.28191
Critical values	$\frac{ \sqrt{n}R_{n,0.5} }{\text{MED}}$	1.11926	1.04216	1.02376	1.01520	1.01877	1.01625	1.01765	1.01196	1.01192	1.01263	1.01124	1.01582	1.00977	1.01294	1.01232	1.01366	1.01748	1.01063	1.00732	1.01072	1.01841	1.01440	1.00791	1.01192	1.01903	1.01314
Table 13:	$\frac{ \sqrt{n}R_{n,0.5} }{MLE}$	1.06865	1.03597	1.02705	1.03060	1.02478	1.01301	1.01447	1.02122	1.01443	1.01287	1.02219	1.01462	1.02344	1.01403	1.01313	1.01834	1.01740	1.01342	1.01943	1.01814	1.01645	1.01320	1.01877	1.00953	1.01907	1.01314
	$ \sqrt{n}R_{n,0.2} $ MED	5.90912	5.10804	4.73081	4.56416	4.45201	4.35032	4.33770	4.27661	4.27821	4.26874	4.25523	4.22512	4.24080	4.26364	4.21718	4.23134	4.23655	4.23949	4.25830	4.21437	4.20401	4.18322	4.19560	4.20832	4.20821	4.19818
	$ \sqrt{n}R_{n,0.2} $ MLE	4.17008	4.19955	4.19686	4.18362	4.22461	4.18853	4.20833	4.19136	4.22155	4.19311	4.19972	4.20073	4.19012	4.21490	4.21452	4.19603	4.20765	4.20222	4.20538	4.20693	4.19724	4.16590	4.17603	4.18318	4.20214	4.19818
	и	20	40	09	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	420	440	460	480	500	8

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Table 14: Critical values of $ \sqrt{n}R_{n,a} $ statistic, for N=100000 repetitions and $\lambda = 5$.	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	<u>6027</u> 1.11188 0.29661 0.27925 0.07042 0.06805 0.00866 0.00873	100 1.04123 0.28822 0.27920 0.06883 0.06771 0.00863 0.00865	(461 1.03082 0.28754 0.28193 0.06893 0.06796 0.00864 0.00866	2995 1.01911 0.28610 0.28061 0.06839 0.06773 0.00864 0.00865	103 1.02076 0.28421 0.28053 0.06810 0.06766 0.00865 0.00866	(439 1.01705 0.28459 0.28094 0.06815 0.06774 0.00859 0.00863	741 1.01490 0.28483 0.28138 0.06808 0.06778 0.00861 0.00862	510 1.01972 0.28409 0.28297 0.06825 0.06798 0.00856 0.00856	172 1.00914 0.28201 0.27947 0.06775 0.06744 0.00860 0.00858	(439 1.01950 0.28520 0.28248 0.06827 0.06797 0.00860 0.00860	-456 1.01674 0.28444 0.28273 0.06814 0.06795 0.00859 0.00861	258 1.01575 0.28307 0.28180 0.06802 0.06759 0.00861 0.00861	933 1.01740 0.28356 0.28193 0.06816 0.06807 0.00863 0.00864	446 1.01433 0.28403 0.28201 0.06813 0.06787 0.00864 0.00864	919 1.01230 0.28236 0.28172 0.06796 0.06773 0.00860 0.00860	789 1.01045 0.28301 0.28149 0.06763 0.06745 0.00859 0.00860	672 1.01201 0.28376 0.28254 0.06790 0.06776 0.00863 0.00863	395 1.01285 0.28265 0.28261 0.06794 0.06786 0.00860 0.00862	158 1.01630 0.28400 0.28305 0.06799 0.06799 0.00861 0.00860	185 1.01579 0.28337 0.28247 0.06788 0.06772 0.00862 0.00861	258 1.01327 0.28118 0.28027 0.06743 0.06736 0.00860 0.00860	509 1.01606 0.28344 0.28199 0.06772 0.06769 0.00857 0.00858	400 1.01221 0.28236 0.28124 0.06810 0.06802 0.00864 0.00865	868 1.01932 0.28370 0.28290 0.06804 0.06784 0.00858 0.00859	916 1.01679 0.28350 0.28182 0.06787 0.06770 0.00862 0.00861	
stic, for N=100	$R_{n,1} \sqrt{n}R$	0.070	0.0688	l <u>9</u> 3 0.0689	0.068	0.068	94 0.068	138 0.0680	297 0.0682	947 0.067	248 0.0682	273 0.068	0.068([<u>93</u> 0.068]	201 0.068	172 0.0679	[49 0.067(254 0.0679	261 0.0679	305 0.0679	247 0.0678	0.067	0.067	0.068	0.0680	182 0.0678	
$ \sqrt{n}R_{n,a} $ statis	$\overline{n}R_{n,1} $ $ \sqrt{n}$	9661 0.279	8822 0.279	8754 0.28	8610 0.280	8421 0.280	8459 0.280	8483 0.28	8409 0.282	8201 0.279	8520 0.282	8444 0.282	8307 0.28]	8356 0.281	8403 0.282	8236 0.281	8301 0.281	8376 0.282	8265 0.282	8400 0.283	8337 0.282	8118 0.280	8344 0.28	8236 0.28]	8370 0.282	8350 0.281	
itical values of	$\overline{AB}_{n,0.5} $ $ \sqrt{MI}$	11188 0.2	04123 0.2	03082 0.2	01911 0.2	02076 0.2	01705 0.2	01490 0.2	01972 0.2	00914 0.2	01950 0.2	01674 0.2	01575 0.2	01740 0.2	01433 0.2	01230 0.2	01045 0.2	01201 0.2	01285 0.2	01630 0.2	01579 0.2	01327 0.2	01606 0.2	01221 0.2	01932 0.2	01679 0.2	
Table 14: Cr	$\frac{ \sqrt{n}R_{n,0.5} }{MLE} \overset{ }{M}$	1.06027 1.	1.04100 1.	1.03461 1.	1.02995 1.	1.02103 1.	1.02439 1.	1.01741 1.	1.02510 1.	1.01172 1.	1.02439 1.	1.02456 1.	1.02258 1.	1.01933 1.	1.01446 1.	1.01919 1.	1.01789 1.	1.01672 1.	1.01395 1.	1.02158 1.	1.02185 1.	1.01258 1.	1.01509 1.	1.01400 1.	1.01868 1.	1.01916 1.	
	$\left \sqrt{n}R_{n,0.2} ight $ MED 1	5.83832	5.03037	4.68703	4.51665	4.43083	4.35525	4.31838	4.31109	4.25517	4.28196	4.26522	4.25199	4.24382	4.20102	4.22986	4.21130	4.19939	4.21324	4.24101	4.21522	4.22878	4.19435	4.22726	4.22241	4.22146	
	$ \sqrt{n}R_{n,0.2} $ MLE	4.15862	4.18722	4.19557	4.20230	4.19353	4.20346	4.17824	4.22484	4.19214	4.21680	4.20422	4.20907	4.22791	4.19666	4.20267	4.17988	4.18848	4.18101	4.21101	4.20429	4.21409	4.19203	4.19923	4.20922	4.20593	
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14: Critical values of $ \sqrt{n}R_{n,a} $	
Table 14: Critical values of $ \sqrt{n}R_{n,a} $	

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