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THE RELATION BETWEEN PAPPUS'S AND CEVA'S THEOREM

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Abstract. In this paper, we provide a proof of Pappus's theorem following the idea presented in the book "Geometry Revisited" by H.S.M. Coxeter and S.L. Greitzer. Our proof is based on Ceva's theorem instead of Menelaus's theorem.

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Introduction

Pappus of Alexandria, who lived in the 4^{th} century A.D., was one of the last great geometers of antiquity. His major work in geometry is entitled *Synagogue* or the *Mathematical Collection* and was originally written in eight books. In it, he collected various discoveries of the most famous mathematicians and a multitude of curious propositions and lemmas intended to facilitate reading their works. Still, the collection includes several important discoveries made by him. Two propositions of *Book VII* (Propositions 138 and 139, see [3]), have together become known as *Pappus's theorem*, the theorem that we will discuss here.

Pappus proved his theorem using Euclidean methods – proportion. Throughout the centuries, this theorem has inspired mathematicians, and consequently, nowadays, there are plenty of different proofs of Pappus's theorem. Some involve the use of a famous theorems, and some others use vector algebra, projective geometry, symbolic computation, computer programs, etc.

Although the proof of Pappus's theorem can be found in a multitude of books, a view from a different angle is always valuable, so we intend to present another exciting proof of this remarkable theorem. Let us start by recalling the proof of Pappus's Theorem given in the brilliant book "Geometry Revisited" by H. S. M. Coxeter and S. L. Greitzer. Their proof was based on Menelaus's Theorem (see [1], p. 67). Precisely, they proved Pappus's Theorem by applying Menelaus's Theorem six times. Following the idea presented in that book, we intend to reveal the connection between Pappus's and Ceva's theorem.

In Section 1 we shall say a few words about Ceva's theorem. In addition, we will formulate Pappus's theorem and point out where several different proofs may be found. In Section 2 we will give a proof of Pappus's theorem, which illuminates its connection with Ceva's theorem.

1. Preliminaries

Ceva's Theorem

Ceva's theorem gives a necessary and sufficient condition which ensures that the line segments joining a vertex of a triangle to a point on the opposite side are concurrent. As is known, these segments are called cevians. More precisely, if X, Y, Z are points on the respective sides BC, CA, AB of triangle ABC, the segments AX, BY, CZ are cevians. This term comes from the name of the Italian mathematician Giovanni Ceva (1647–1734), who first published proof of the mentioned theorem [1]. Although he was not the first mathematician who knew about it[1], this theorem now bears his name. We will not prove this classical theorem here. Instead, we refer the readers to the book [1] where the proof may be found.

Theorem 1. (Ceva's Theorem) *Let ABC be a triangle. The cevians AX, BY, CZ are concurrent if and only if the product of three oriented length ratios along each of the triangle edges satisfies the equality*

$$\frac{|AZ|}{|ZB|}\frac{|BX|}{|XC|}\frac{|CY|}{|YA|} = 1.$$

Remark 1. By $|AB|$ we denote the distance from a point A to a point B. For three distinct collinear points P, Q, R we set

$$\frac{|PR|}{|RQ|} := \begin{cases} \frac{PR}{RQ}, & \text{if point } R \text{ is between } P \text{ and } Q, \\ -\frac{PR}{RQ}, & \text{otherwise.} \end{cases}$$

Under the term *Ceva triangle*, we assume a triangle with the points X, Y, Z on the edges BC, CA, AB, respectively, such that

$$\frac{|AZ|}{|ZB|}\frac{|BX|}{|XC|}\frac{|CY|}{|YA|} = 1.$$

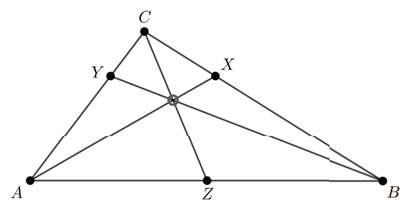

Fig. 1. Ceva triangle

Furthermore, we will call the points X, Y, Z the *auxiliary points* and the point of intersection of cevians will be called the *Ceva point*.

Pappus's theorem

Every student sooner or later becomes familiar with the famous Pappus's theorem. This well-known theorem, as one of the most important theorems of plane geometry, was a subject of many investigations and inspired a lot of fascinating mathematics. We state it as follows:

[1]This beautiful theorem was actually discovered by Arab mathematician Yusuf Al-Mutaman ibn Hud in the 11th century. [2]

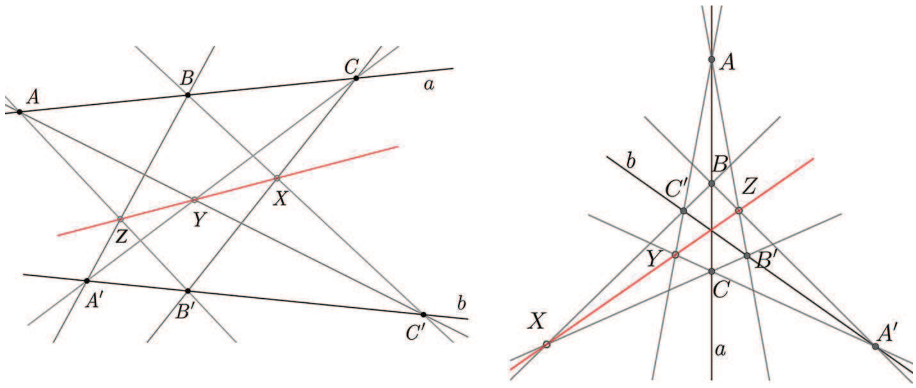

Fig. 2. Two drawings of Pappus's Theorem

Theorem 2. (Pappus's Theorem) *If A, B, C are three points on one line and A', B', C' on another, then the three points of intersection $X = BC' \cap B'C$, $Y = AC' \cap A'C$ and $Z = AB' \cap A'B$ are collinear.*

The interested reader can find information on ancient methods of proof in [5]. The original proof can be found in [3]. In addition to the mentioned books, we would like to point out the beautiful book on projective geometry by J. Richter-Gebert [4] where one can find various proofs.

2. Proof of Pappus's Theorem using Ceva's Theorem

In Figure 2, we presented two drawings of Pappus's theorem. The basic idea of the following proof is to observe the right drawing and to look for Ceva relations on it. For the labelling in this proof, we refer to Figure 3.

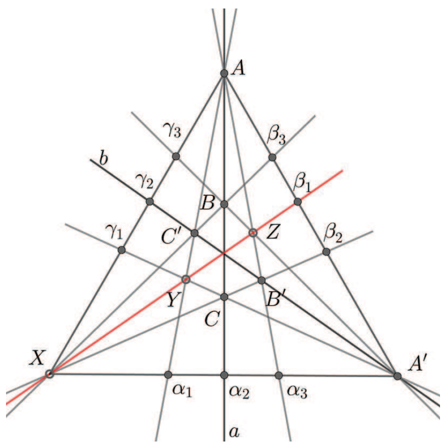

Fig. 3. Labelling for the proof

Proof. We start by considering the triangle $XA'A$. Our aim is to prove that the points X, Y and Z are collinear. The idea for proving this is elementary. We suppose that XY intersects AA' at the point β_1 and prove that XZ will intersect AA' at the same point. For this, let us take a closer look at Figure 3.

Applying Ceva's theorem to triangle $XA'A$ and point Y, i.e. using the concurrence of cevians $A\alpha_1$, $X\beta_1$, $A'\gamma_1$, we get

$$\frac{|X\alpha_1|}{|\alpha_1 A'|}\frac{|A'\beta_1|}{|\beta_1 A|}\frac{|A\gamma_1|}{|\gamma_1 X|} = 1. \tag{1}$$

Analogously, if we consider the same triangle and points C, B', C' and B, respectively, we obtain four more Ceva's relations

$$\frac{|X\gamma_1|}{|\gamma_1 A|}\frac{|A\beta_2|}{|\beta_2 A'|}\frac{|A'\alpha_2|}{|\alpha_2 X|} = 1 \quad (2) \qquad \frac{|A'\beta_2|}{|\beta_2 A|}\frac{|A\gamma_2|}{|\gamma_2 X|}\frac{|X\alpha_3|}{|\alpha_3 A'|} = 1 \quad (3)$$

$$\frac{|X\gamma_2|}{|\gamma_2 A|}\frac{|A\beta_3|}{|\beta_3 A'|}\frac{|A'\alpha_1|}{|\alpha_1 X|} = 1 \quad (4) \qquad \frac{|A'\beta_3|}{|\beta_3 A|}\frac{|A\gamma_3|}{|\gamma_3 X|}\frac{|X\alpha_2|}{|\alpha_2 A'|} = 1. \quad (5)$$

Multiplying all these relations and canceling terms that occur in the numerator as well as in the denominator, we are led to the relation

$$\frac{|A'\beta_1|}{|\beta_1 A|}\frac{|A\gamma_3|}{|\gamma_3 X|}\frac{|X\alpha_3|}{|\alpha_3 A'|} = 1. \tag{6}$$

This translates to the fact that the cevians $A\alpha_3$, $X\beta_1$, $A'\gamma_3$ are concurrent, and since $A\alpha_3$ and $A'\gamma_3$ intersect in a point Z, that is the Ceva point in this case. In other words, we can deduce that XZ intersects AA' in the point β_1, just as the segment XY. Thus we are done. ∎

Here an interesting construction can be viewed. Let us imagine glueing Ceva's triangles corresponding to relations (1) and (2). For example, the first triangle is coloured yellow and the second pink. We will glue them along the edge AX, since they have the same auxiliary point on it (this holds because A', C, Y are collinear points). Figure 4 illustrates this. What do we get out of it?

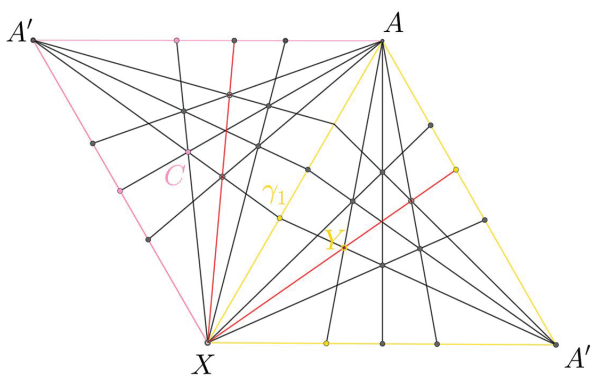

Fig. 4. Gluing the first and the second Ceva triangles

Well, if we multiply the left-hand sides and the right-hand sides of the relations (1) and (2), we end up with the relation

$$\frac{|X\alpha_1|}{|\alpha_1 A'|}\frac{|A'\beta_1|}{|\beta_1 A|}\frac{|A\beta_2|}{|\beta_2 A'|}\frac{|A'\alpha_2|}{|\alpha_2 X|} = 1. \tag{7}$$

More precisely, the ratio on the common edge cancels, and we get the relation between ratios corresponding to a boundary of the quadrilateral formed by these two triangles. Thus, we can conclude that if we glue two Ceva triangles along an edge in a way that they share the auxiliary point on it, we will end up with an expression that contains only ratios from the boundary of the resulting quadrilateral. Moreover, continuing in the same fashion, we obtain a collection of Ceva triangles glued along edges and the relation between ratios corresponding to the boundary of the figure formed by these triangles.

It turns out that by applying this procedure, we may, without any problem, prove Pappus's theorem. To be a little more concrete, note that we found three more Ceva triangles, those corresponding to the relations (3), (4) and (5). For example, the first triangle is coloured blue, the second purple and the third green. One after the other, let us glue them edge to edge in a proper way. Figure 5 shows this process.

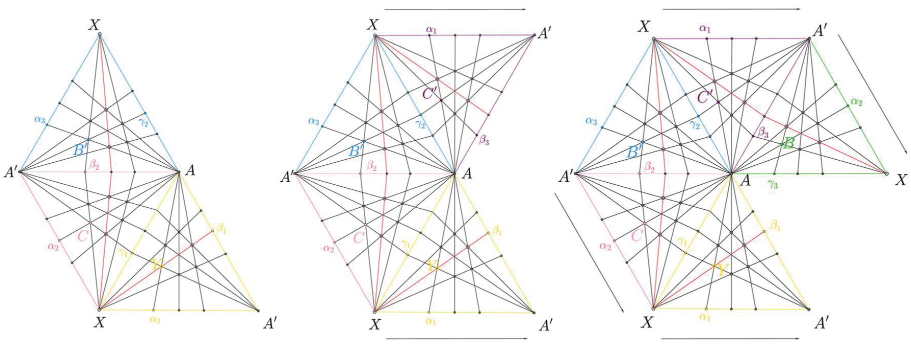

Fig. 5. Gluing Ceva triangles

Gluing the blue triangle and considering the product of the relation (3) obtained from it and the relation (7) leaves us with

$$\frac{|X\alpha_1|}{|\alpha_1 A'|}\frac{|A'\beta_1|}{|\beta_1 A|}\frac{|A\gamma_2|}{|\gamma_2 X|}\frac{|X\alpha_3|}{|\alpha_3 A'|}\frac{|A'\alpha_2|}{|\alpha_2 X|} = 1. \tag{8}$$

The left picture in Figure 5 illustrates this. What may be seen at the first glance is that gluing the blue triangle along the edge $A'A$ to the pink one was easy. On the other hand, we must pay special attention to the purple and the green triangle.

Consider, in particular, the case of the purple triangle. Note that not only do the purple and the blue triangle have a common edge, but the purple and the yellow one has it too (in Figure 5 in the middle, we represent the gluing of their edges by arrows). In this regard, if we take into account the relation (4) obtained from the purple triangle as well as the relation (8), and consider their product, after canceling the ratios on the common edges, we get

$$\frac{|A'\beta_1|}{|\beta_1 A|}\,\frac{|A\beta_3|}{|\beta_3 A'|}\,\frac{|A'\alpha_2|}{|\alpha_2 X|}\,\frac{|X\alpha_3|}{|\alpha_3 A'|} = 1. \tag{9}$$

What is left is to add a green triangle to the drawing (see the right picture in Figure 5). Hence, let us multiply the left and the right sides of the previous relation and the relation (6) implied by the green triangle. In the end, we are left with

$$\frac{|A'\beta_1|}{|\beta_1 A|}\,\frac{|A\gamma_3|}{|\gamma_3 X|}\,\frac{|X\alpha_3|}{|\alpha_3 A'|} = 1,$$

which is the same relation as the relation (6).

Finally, we see that the relation (6) could be derived without calculation. As mentioned previously, if we glue several Ceva triangles along common edges (note that they need to share the auxiliary point on it), we get the relation corresponding to the boundary of the resulting figure. In this case, that is a triangle! So, using the final figure, we could immediately conclude that we will end up with the relation (6). The right picture in Figure 5 illustrates this.

In Figure 6, we presented the missing triangle to form the hexagon in Figure 5 (right), i.e. the hexagon in Figure 7. That is the Ceva triangle that corresponds to the relation (6).

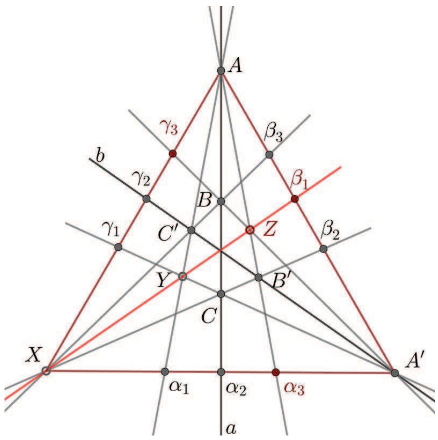

Fig. 6. The last Ceva triangle

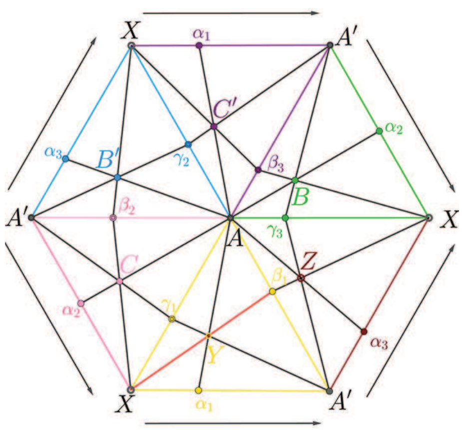

Fig. 7. X, Y and Z are collinear

Come to an end, we encourage the readers to compare the proof of Pappus's theorem given in [1] with the proof presented in this paper.

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