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Norm Estimates for Remainders of Noncommutative Taylor Approximations for Laplace Transformers Defined by Hyperaccretive Operators

Danko R. Jocić

Department of Mathematics, University of Belgrade, Studentski trg 16, P.O. Box 550, 11000 Belgrade, Serbia; danko.jocic@matf.bg.ac.rs

Abstract: Let $\mathcal{H}$ be a separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, μ a finite Borel measure on $\mathbb{R}_+$ with the finite $(n+1)$-th moment, $f(z) := \int_{\mathbb{R}_+} e^{-tz} d\mu(t)$ for all $\Re z \geqslant 0, \mathcal{C}_\Psi(\mathcal{H})$, and $||\cdot||_\Psi$ the ideal of compact operators and the norm associated to a symmetrically norming function Ψ, respectively. If $A, D \in \mathcal{B}(\mathcal{H})$ are accretive, then the Laplace transformer on $\mathcal{B}(\mathcal{H})$, $X \mapsto \int_{\mathbb{R}_+} e^{-tA} X e^{-tD} d\mu(t)$ is well defined for any $X \in \mathcal{B}(\mathcal{H})$ as is the newly introduced Taylor remainder transformer $R_n(f; D, A)X := f(A)X - \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} X D^i f^{(k)}(D)$. If A, D^* are also $(n+1)$-accretive, $\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} A^{n+1-k} X D^k \in \mathcal{C}_\Psi(\mathcal{H})$ and $||\cdot||_\Psi$ is Q* norm, then $||\cdot||_\Psi$ norm estimates for $\left(\sum_{k=0}^{n+1} \binom{n+1}{k} A^k A^{n+1-k}\right)^{\frac{1}{2}} R_n(f; D, A) X \left(\sum_{k=0}^{n+1} \binom{n+1}{k} D^{n+1-k} D^{*k}\right)^{\frac{1}{2}}$ are obtained as the spacial cases of the presented estimates for (also newly introduced) Taylor remainder transformers related to a pair of Laplace transformers, defined by a subclass of accretive operators.

Keywords: norm inequalities; Q and Q-norms; n-(hyper)accretive operators

MSC: 47B49; 47B47; 47B44; 47A56; 47A30; 47A63; 47B10; 47B15

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1. Introduction

The widely known type of the remainder of Taylor approximations

$$R_p(\varphi; H_\circ, V) \stackrel{\text{def}}{=} \varphi(H_\circ + V) - \sum_{k=0}^{p} \frac{1}{k!} \frac{d^k}{(dx)^k} \varphi(H_\circ + xV)|_{x=0} \tag{1}$$

for $p \in \mathbb{N}, V = V^* \in \mathcal{C}_p(\mathcal{H})$, self-adjoint $H_\circ$ acting in a Hilbert space $\mathcal{H}$ and a wide class of functions φ was (among others) considered by Koplienko in [1], Neindhardt in [2], Dostanić in [3], Peller in [4], and Skripka in [5], where the sufficient conditions for the existence of higher-order Lifshits–Krein spectral shift functions, the trace norm estimate, and the corresponding trace formula for $R_p(\varphi; H_\circ, V)$ were presented. Further generalizations and other variants of trace formulas for the remainders of Taylor approximations can be found in [6–10] and the references therein. For the use of multilinear operator integrals in the representation of the remainders of Taylor approximations, see [11,12].

Operator-valued Laplace transforms and Laplace transformers in norm ideals of compact operators were introduced in [13]. They soon proved useful in deriving the extension of the arithmetic–geometric (A-G) mean, Young's norm inequalities to accretive operators in [14], and higher-order A-G mean inequalities for hyperaccretive subnormal operators in [15]. It turned out that the hyperaccretivity plays a natural and crucial role in those inequalities. Furthermore, the iterated perturbation norm inequalities for Laplace transformers induced by accretive operator families were studied in [16], where the special cases of (h4), (h5) of Ineq. (4.4); (h7) and (h8) of Ineq. (4.5); and (h10)–(h12) of Ineq. (4.6) of the noncommutative Pick–Julia theorems for generalized derivations in [17] were generalized to perturbations of Laplace transformers. See [16] (Th. 3.8, Th. 3.1, Th. 3.3).

The motivation for this paper is the introduction of the two new and natural (alternative) noncommutative Tailor remainders, together with the demonstration of their suitability and usefulness in answering some classical problems in operator theory and matrix analysis. Here, we focus on the establishing norm estimates for those new types of remainders of Taylor approximations introduced by Definition 3, which are not necessarily the trace class operators. In addition to the generalized derivation norm inequalities for operators, the results obtained in this paper also include norm inequalities for perturbations of Laplace transformers, which further develop the case $n := 1$ of the results presented in [16]. The most important tools for obtaining these results are the Cauchy–Schwarz norm Ineq. (23) in [18] (Th. 3.2.), Ineq. (30) in [19] (Th. 3.1 (c)), the first inequality in (32) in [19] (Th. 3.1 (d)) and Ineq. (32) in [19] (Th. 3.1 (e)), which were previously used to derive the results presented in [13–16]. We therefore strongly advise interested readers to inform themselves about these inequalities before reading the "Main results" section in this article.

The class of n-hypercontractions was first introduced by Agler in [20,21]. There is a strong parallelism between classes of hyperaccretive and hypercontractive operators, and the Cayley transform on accretive and contractive operators represents a very important tool to correlate their properties; for examples, see [15] (Lemma 3.4(d)). For any $n \in \mathbb{N}$, let $\boldsymbol{A}_n(\mathbb{D})$ be the Hilbert space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}\colon z \mapsto \sum_{k=0}^{\infty} c_k z^k$, satisfying $||f||^2_{\boldsymbol{A}_n} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1}^{-1} |c_k|^2 < +\infty$. The backward shifts on the Bergman spaces are not only the best known examples of n-hypercontractive operators but also model operators for n-hyperacontractive operators such that they are of paramount importance in the world of operator-related function theory. For more about Bergman spaces, see [22].

2. Notation

In this paper, we denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on a separable, complex Hilbert space $\mathcal{H}$. For a symmetrically norming (s.n.) function Φ, defined on sequences of complex numbers, there is the corresponding symmetric or a unitary invariant (u.i.) norm $||\cdot||_\Phi$ on operators, defined on the naturally associated norm ideal $\mathcal{C}_\Phi(\mathcal{H}) \subset \mathcal{C}_\infty(\mathcal{H})$, where $\mathcal{C}_\infty(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$. Thus, the Schatten–von Neumann trace ideals $\mathcal{C}_p(\mathcal{H})$ for $1 \leqslant p \leqslant +\infty$ are associated to ℓ^p s.n. functions, given by $\ell^p\big((\lambda_n)_{n=1}^\infty\big) \stackrel{\text{def}}{=} \sqrt[p]{\sum_{n=1}^\infty |\lambda_n|^p}$. Schatten p-norms are classical examples of (degree) p-modified norms. In fact, any u.i. norm $||\cdot||_\Phi$ can be p-modified for any $p > 0$ by setting $||A||_{\Phi^{(p)}} \stackrel{\text{def}}{=} ||\,|A|^p||_\Phi^{1/p}$ for all $A \in \mathcal{C}_\infty(\mathcal{H})$, such that $|A|^p \in \mathcal{C}_\Phi(\mathcal{H})$. From now on, we refer to s.n. function $\Phi^{(p)}$ as a p-modification of the s.n. function Φ. For the elementary proof of the triangle inequality for $p \geqslant 1$ and other properties of those norms, including the Hölder inequality, see the preliminary section in [23] or [24] (cor. IV.2.6, ex. IV.2.7-8). Hence, $\mathcal{C}_p(\mathcal{H}) = \mathcal{C}_{\ell^p}(\mathcal{H})$, and its norm is simply denoted by $||\cdot||_p$.

The following useful monotonicity property for u.i. norms, which states that

$$||AXB||_\Phi \leqslant ||CXD||_\Phi, \tag{2}$$

whenever $A^*A \leqslant C^*C$ and $BB^* \leqslant DD^*$, will be needed later on. For the proof of (2) see [25] (p. 62).

Also, for each s.n. function Φ, there is the adjoint s.n. function, which is denoted by Φ^*.

In order to obtain a more comprehensive insight into the theory of norm ideals, the reader is referred to [24,26–28].

If $(\Omega, \mathfrak{M}, \mu)$ is a space Ω with a measure μ on σ-algebra $\mathfrak{M}$, consisting of (measurable) subsets of Ω, then we call a function $A\colon \Omega \to \mathcal{B}(\mathcal{H})\colon t \mapsto A_t$ $(\mathfrak{M})$ *weakly* measurable* if $t \mapsto \langle A_t g, h\rangle$ is a $(\mathfrak{M})$ is measurable for all $g, h \in \mathcal{H}$. If these functions are also $[\mu]$ integrable on Ω, then A is called $([\mu])$ *weakly* integrable* on Ω, and in this case for all $\delta \in \mathfrak{M}$, there is the unique (known as Gel'fand or *weak* integral*, which is denoted by ${}^{w^*}\!\!\int_\delta A_t\,d\mu(t)$) operator in $\mathcal{B}(\mathcal{H})$ that satisfies (amongst others)

$$\left\langle \overset{w^*}{\int_\delta} A_t\,d\mu(t)g,h \right\rangle = \int_\delta \langle A_t g,h\rangle\,d\mu(t) \qquad \text{for all } g,h\in\mathcal{H}.$$

For a more detailed insight into the weak*-integrability of operator valued (o.v.) functions, the reader is referred to [29] (p. 53), [18] (p. 320) and [30] (lemma 1.2). Let also $L^2_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ denote the space of all weakly* measurable functions $A\colon\Omega\to\mathcal{B}(\mathcal{H})$ such that $\int_\Omega \|A_t h\|^2\,d\mu(t)<+\infty$ for any $h\in\mathcal{H}$, which we call the space of ($[\mu]$) *square integrable* (*s.i.*) functions. Note that o.v. function $t\mapsto A_t^*A_t$ are Gel'fand integrable if and only if $A\in L^2_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ as it was shown in [18] (ex. 2).

From this point on, we will use the simplified notation $\int_\delta A_t\,d\mu(t)$ instead of ${}^{w^*}\!\!\int_\delta A_t\,d\mu(t)$. For a family $\{A_t\}_{t\in\Omega}$ of mutually commutative normal operators, use the acronym: *m.c.n.o.* (family). For $A,B\in\mathcal{B}(\mathcal{H})$, the *bilateral multiplier* $A\otimes B$ is defined by $A\otimes B\colon\ \mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})\colon X\mapsto AXB$ and the *generalized derivation* $\Delta_{A,B}$ by $\Delta_{A,B}\overset{\text{def}}{=}A\otimes I+I\otimes B$. If $A,B\colon\Omega\to\mathcal{B}(\mathcal{H})$ are weak*-measurable such that $t\mapsto A_tXB_t$ is weak*-integrable on Ω for all $X\in\mathcal{C}_\Phi(\mathcal{H})$, then $\int_\Omega A_t\otimes B_t\,d\mu(t)\colon\mathcal{C}_\Phi(\mathcal{H})\to\mathcal{B}(\mathcal{H})\colon X\mapsto\int_\Omega A_tXB_t\,d\mu(t)$ is called the *inner product type* (*i.p.t.*) *transformer* on $\mathcal{C}_\Phi(\mathcal{H})$ if $\int_\Omega A_tXB_t\,d\mu(t)\in\mathcal{C}_\Phi(\mathcal{H})$ for all $X\in\mathcal{C}_\Phi(\mathcal{H})$. For the existence and different types of convergence for weak*-integrals, as well as for the boundedness of i.p.t. transformers, the reader is referred to [18] (lemma 3.1, th. 3.2, th. 3.3, th. 3.4).

The concept of Gel'fand integrability for o.v. functions and the associated i.p.t. transformers had proved to be very fruitful, and, in particular, has led to the emergence of a wide range of quite new Cauchy–Schwarz operator and norm inequalities for i.p.t. transformers in [18,19,31]. These inequalities have enabled (among other things) the spectral measures free approach to double operator integrals (DOI), developed by Birman, Solomyak and their collaborators, which has served as the main tool for perturbation and derivation inequalities for functions of normal operators, mainly self-adjoint and unitary operators, as well as for the treatment of new classes of operators, including N-hyperaccretive, N-hypercontractive, quasinormal, hyponormal, subnormal, and operators with the contractive real part. The results obtained are presented in [13–17,25,32–34] and other related papers.

Next, we recall the definitions of some important subclasses of bounded operators on Hilbert spaces, which are discussed below.

Definition 1. *For operators $A\in\mathcal{B}(\mathcal{H})$ and $n,N\in\mathbb{N}$ we say the following:*

1. *A is accretive if $A^*+A\geqslant 0$;*
2. *A is strictly (or uniformly) accretive if there is a constant $c>0$, such that $A^*+A\geqslant cI$, which will be denoted by $A^*+A\gg 0$;*
3. *A is N-accretive if, and only if, $\sum_{k=0}^n\binom{N}{k}A^{*k}A^{N-k}\geqslant 0$;*
4. *A is strictly (or uniformly) N-accretive if, and only if, $\sum_{k=0}^n\binom{N}{k}A^{*k}A^{N-k}\gg 0$, i.e., if $\sum_{k=0}^n\binom{N}{k}A^{*k}A^{N-k}\geqslant cI$ for some $c>0$;*
5. *A is N-semiaccretive if, and only if, $\sum_{k=0}^n\binom{N}{k}A^{*k}A^{N-k}$ semidefinite;*
6. *A is N-hyperaccretive if, and only if it is n-accretive for all $1\leqslant n\leqslant N$;*

More about the importance of accretive operators in the study of the stability of solutions of differential equations in Banach space can be seen in [35].

Throughout this paper, we use the conventions $\amalg_+\overset{\text{def}}{=}\mathbb{C}_+\overset{\text{def}}{=}\{z\in\mathbb{C}\colon\Re z\overset{\text{def}}{=}\frac{z+\bar z}{2}>0\}$, $\mathbb{R}_+\overset{\text{def}}{=}[0,+\infty)$, as well as $\mathbb{Z}$ for the set of integers, $\mathbb{N}\overset{\text{def}}{=}\mathbb{Z}\cap[1,+\infty)$ and $\mathbb{N}_\circ\overset{\text{def}}{=}\mathbb{Z}\cap[0,+\infty)$.

We must also emphasize that we always cite (address to) the unnumbered line in a multiline formula as (to) a part of the following numbered one.

3. Preliminaries and Preparatory Results

3.1. General Preliminaries

Definition 2. *Let μ be a Borel measure on $\mathbb{R}_+$ and $A, B, X \in \mathcal{B}(\mathcal{H})$. If a function $t \mapsto e^{-tA}$ is Gel'fand $[\mu]$-integrable on $\mathbb{R}_+$, then*

$$\mathscr{L}[\mu](A) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} e^{-tA}\, d\mu(t)$$

will denote the operator valued **(***o.v.***)** *Laplace transform of μ (evaluated in A), and similarly, if the function $t \mapsto e^{-tA}Xe^{-tB}$ is Gel'fand $[\mu]$-integrable on $\mathbb{R}_+$, then*

$$\mathscr{L}[\mu]\Delta_{A,B}X \stackrel{\text{def}}{=} \mathscr{L}[\mu](\Delta_{A,B})X \stackrel{\text{def}}{=} \mathscr{L}[\mu](A\otimes I + I\otimes B)X \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} e^{-tA}Xe^{-tB} d\mu(t)$$

will denote the Laplace transformer of μ (in a generalized derivation $A\otimes I + I\otimes B$, evaluated in X). In this case, we say that X in the domain of $\mathscr{L}[\mu]\Delta_{A,B}$ and we denote this by $X \in \mathscr{D}_{\mathscr{L}[\mu]\Delta_{A,B}}$.

Example 1. *If $A, B \in \mathcal{B}(\mathcal{H})$ are accretive, $X \in \mathcal{B}(\mathcal{H})$, and μ is a finite Borel measure on $\mathbb{R}_+$, then $\mathscr{D}_{\mathscr{L}[\mu]\Delta_{A,B}} = \mathcal{B}(\mathcal{H})$. Namely,*

$$\int_{\mathbb{R}_+} e^{-tA}e^{-tA^*} d\mu(t) \leqslant \int_{\mathbb{R}_+} I\, d\mu(t) = \mu(\mathbb{R}_+)I \quad \text{and} \quad \int_{\mathbb{R}_+} e^{-tB^*}e^{-tB}\, d\mu(t) \leqslant \mu(\mathbb{R}_+)I,$$

so $\{\, e^{-tA^*} \,\}_{t\geqslant 0}$ and $\{\, e^{-tB} \,\}_{t\geqslant 0}$ are $[\mu]$ s.i. families, so $\mathscr{L}\,[\mu]\Delta_{A,B}X = \int_{\mathbb{R}_+} e^{-tA}Xe^{-tB}\, d\mu(t)$ is well defined and it satisfies $||\mathscr{L}[\mu]\Delta_{A,B}X|| \leqslant \mu(\mathbb{R}_+)||X||$ based on the estimate (12) in [18] (Lemma 3.1(a)).

Also, if $N \in \mathbb{N}$, then for all $X \in \mathcal{B}(\mathcal{H})$

$$\tfrac{1}{(N-1)!}\int_{\mathbb{R}_+} t^{N-1}e^{-tA}\Delta^N_{A,B}Xe^{-tB}dt = X, \qquad \text{if } A \text{ or } B \text{ is strictly accretive, and} \tag{3}$$

$$\tfrac{1}{(N-1)!}\int_{\mathbb{R}_+} t^{N-1}e^{-tA^*}\Delta^N_{A^*,A}(I)e^{-tA}dt = I - \mho^-_{A^*,A}(I) \stackrel{\text{def}}{=} I - \operatorname*{{}^s\!\lim}_{t\to+\infty} e^{-tA^*}e^{-tA}, \tag{4}$$

if A is N-hyperaccretive. According to Formula (5) in [14] (Lemma 2.4) for all $T \in \mathbb{R}_+$

$$\tfrac{1}{(N-1)!}\int_{[0,T]} t^{N-1}e^{-tA}\Delta^N_{A,B}Xe^{-tB}dt = X - \sum_{n=0}^{N-1} \tfrac{T^n}{n!}e^{-TA}\Delta^n_{A,B}Xe^{-TB}, \tag{5}$$

where ${}^s\!\lim_{T\to+\infty} T^ne^{-TA} = 0$ or ${}^s\!\lim_{T\to+\infty} T^ne^{-TB} = 0$, based on the estimate $||T^ne^{-TA}h|| \leqslant T^ne^{-\frac{cT}{2}}||e^{-T(A-\frac{c}{2}I)}h|| \to 0$ as $T \to +\infty$ if A is uniformly accretive satisfying $A + A^* \geqslant cI$ for some $c > 0$. If $B + B^* \geqslant cI$, Formula (3) proves similarly.

Formula (4) was recently shown in [32] (Th. 2.4(7)).

Lemma 1. *For all $t, T \in \mathbb{R}_+$ and $A, B, C, D, X \in \mathcal{B}(\mathcal{H})$ satisfying $AC = CA$ and $BD = DB$*

$$\begin{aligned} e^{tA}Xe^{tB} - e^{tC}Xe^{tD} &= \int_{[0,t]} e^{(t-u)A+uC}(AX+XB-CX-XD)e^{uB+(t-u)D}du \\ &= \int_{[0,1]} e^{(1-s)tA+stC}(AX+XB-CX-XD)e^{stB+(1-s)tD}t\,ds, \end{aligned} \tag{6}$$

$$\begin{aligned} &\int_{[0,T]}(e^{-tA}Xe^{-tB} - e^{-tC}Xe^{-tD})d\mu(t) \\ &= \int_{[0,T]}\int_{[0,1]} e^{-(1-s)tA-stC}(CX+XD-AX-XB)e^{-(1-s)tB-stD}t\,ds\,d\mu(t). \end{aligned} \tag{7}$$

Moreover, if $A, B, C-A$ and $D-B$ are also accretive (as are C, D as well) and μ is a finite Borel measure on $\mathbb{R}_+$, then

$$\mathscr{L}[\mu]\Delta_{A^*,A}(I) - \mathscr{L}[\mu]\Delta_{C^*,C}(I) = \operatorname*{{}^{s}\lim}_{T\to+\infty}\int_{[0,T]}\int_{[0,1]} e^{-(1\text{-}s)tA^*-stC^*}(C^*+C-A^*-A)e^{-(1\text{-}s)tA-stC}t\,ds\,d\mu(t)$$
$$= \int_{\mathbb{R}_+}\int_{[0,1]} e^{-(1\text{-}s)tA^*-stC^*}(C^*+C-A^*-A)e^{-(1\text{-}s)tA-stC}t\,ds\,d\mu(t), \tag{8}$$
$$\mathscr{L}[\mu]\Delta_{B,B^*}(I) - \mathscr{L}[\mu]\Delta_{D,D^*}(I) = \int_{\mathbb{R}_+}\int_{[0,1]} e^{-(1\text{-}s)tB-stD}(D^*+D-B^*-B)e^{-(1\text{-}s)tB^*-stD^*}t\,ds\,d\mu(t), \tag{9}$$
$$\mathscr{L}[\mu]\Delta_{A,B}X - \mathscr{L}[\mu]\Delta_{C,D}X$$
$$= \operatorname*{{}^{s}\lim}_{T\to+\infty}\int_{[0,T]}\int_{[0,1]} e^{-(1\text{-}s)tA-stC}(CX+XD-AX-XB)e^{-(1\text{-}s)tB-stD}t\,ds\,d\mu(t). \tag{10}$$

In this case, $\mathscr{L}[\mu]\Delta_{A^*,A}(I) - \mathscr{L}[\mu]\Delta_{C^*,C}(I) \geqslant 0$ *and* $\mathscr{L}[\mu]\Delta_{B,B^*}(I) - \mathscr{L}[\mu]\Delta_{D,D^*}(I) \geqslant 0$*, and both families* $\{\sqrt{t}\sqrt{\Delta_{C^*-A^*,C-A}(I)}e^{-(1\text{-}s)tA-stC}\}_{0\leqslant t,\,0\leqslant s\leqslant 1}$ *and* $\{\sqrt{t}\sqrt{\Delta_{D^*-B^*,D-B}(I)}e^{-(1\text{-}s)tB-stD}\}_{0\leqslant t,\,0\leqslant s\leqslant 1}$ *are square integrable on* $\mathbb{R}_+\times[0,1]$.

If additionally $(A-C)X + X(B-D) \in \mathcal{C}_1(\mathcal{H})$*, then*

$$\sqrt{\Delta_{C^*-A^*,C-A}(I)}\big(\mathscr{L}[\mu](A\otimes I+I\otimes B)X - \mathscr{L}[\mu](C\otimes I+I\otimes D)X\big)\sqrt{\Delta_{D-B,D^*-B^*}(I)}$$
$$= \int_{\mathbb{R}_+}\int_{[0,t]}\sqrt{\Delta_{C^*-A^*,C-A}(I)}e^{-(t-s)A-sC}\big((C-A)X+X(D-B)\big)e^{-(t-s)B-sD}\sqrt{\Delta_{D-B,D^*-B^*}(I)}\,ds\,d\mu(t)$$
$$\in \mathcal{C}_1(\mathcal{H}). \tag{11}$$

Moreover, if additionally $\int_{\mathbb{R}_+} t\,d\mu(t) < +\infty$*, then*

$$\big(\mathscr{L}[\mu](A\otimes I+I\otimes B)X - \mathscr{L}[\mu](C\otimes I+I\otimes D)X\big)$$
$$= \int_{\mathbb{R}_+}\int_{[0,t]} e^{-(t-s)A-sC}\big((C-A)X+X(D-B)\big)e^{-(t-s)B-sD}ds\,d\mu(t). \tag{12}$$

Proof. The special case $B := C := 0$ and $X := I$ of Formula (6) follows from

$$e^{tA} - e^{tD} = \big(e^{tA}e^{-tD} - I\big)e^{tD} = \int_{[0,t]}\frac{d}{du}\big(e^{uA}e^{-uD} - I\big)du\,e^{tD} \tag{13}$$
$$= \int_{[0,t]} e^{uA}(A-D)e^{-uD}du\,e^{tD} = \int_{[0,t]} e^{uA}(A-D)e^{(t-u)D}du$$
$$= \int_{[0,t]} e^{(t-u)A}(A-D)e^{uD}du = \int_{[0,1]} e^{(1-s)tA}(A-D)e^{stD}t\,ds, \tag{14}$$

where the second equality in (13) is due to the Newton–Leibniz formula, while the equality in (14) is based on the change in variable $u := st$ for $s \in [0,1]$.

By applying (13) and (14) to $-(A\otimes I + I\otimes B)$ instead of A, to $-(C\otimes I + I\otimes D)$ instead of D, and then by integrating on $[0,T]$, this implies the inequalities in (15):

$$\int_{[0,T]}\big(e^{-tA}\otimes e^{-tB} - e^{-tC}\otimes e^{-tD}\big)\,d\mu(t) = \int_{[0,T]}\big(e^{-t(A\otimes I+I\otimes B)} - e^{-t(C\otimes I+I\otimes D)}\big)\,d\mu(t)$$
$$= \int_{[0,T]}\int_{[0,1]} e^{-(1-s)t(A\otimes I)-(1-s)t(I\otimes B)}\big(C\otimes I + I\otimes D - A\otimes I - I\otimes B\big)e^{-st(C\otimes I)-st(I\otimes D)}t\,ds\,d\mu(t) \tag{15}$$
$$= \int_{[0,T]}\int_{[0,1]} e^{-(1-s)tA-stC}\otimes e^{-(1-s)tB-stD}\big(C\otimes I + I\otimes D - A\otimes I - I\otimes B\big)\,t\,ds\,d\mu(t), \tag{16}$$

while the equality in (16) is a consequence of the commutativity within the transformer family $\{A\otimes I, I\otimes B, C\otimes I, I\otimes D\}$. By evaluating Formula (15) and (16) in an arbitrary $X \in \mathcal{B}(\mathcal{H})$, we explicitly obtain the proof for Formula (7) (which is just the alternative form of (15) and (16)).

To prove the integral representation of Formula (10), it is sufficient to use Formula (7) to realize that

$$\mathscr{L}[\mu]\Delta_{A,B}X - \mathscr{L}[\mu]\Delta_{C,D}(X) = \operatorname*{{}^s\!\lim}_{T\to+\infty}\int_{[0,T]}\left(e^{-tA}Xe^{-tB} - e^{-tC}Xe^{-tD}\right)d\mu(t)$$
$$= \operatorname*{{}^s\!\lim}_{T\to+\infty}\int_{[0,T]}\int_{[0,1]} e^{-(1-s)tA-stC}(CX+XD-AX-XB)e^{-(1-s)tB-stD}\,t\,ds\,d\mu(t), \tag{17}$$

where the first equality in (17) is based on the double application of [19] (Lemma 2.1(b)), first to $\{e^{-tA^*}\}_{t\geqslant 0}$ and $\{e^{-tB}\}_{t\geqslant 0}$, and then to $\{e^{-tC^*}\}_{t\geqslant 0}$ and $\{e^{-tD}\}_{t\geqslant 0}$, with their $[\mu]$ square integrability supported by the arguments in Example 1.

If, in addition, operators $C-A$ and $D-B$ are accretive, then any of the families $\{\sqrt{t}\sqrt{\Delta_{C^*-A^*,C-A}(I)}\,e^{-(1-s)tA-stC}\}_{0\leqslant t,0\leqslant s\leqslant 1}$, $\{\sqrt{t}\sqrt{\Delta_{C-A,C^*-A^*}(I)}\,e^{-(1-s)tA^*-stC^*}\}_{0\leqslant t,0\leqslant s\leqslant 1}$, $\{\sqrt{t}\sqrt{\Delta_{D^*-B^*,D-B}(I)}\,e^{-(1-s)tB-stD}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ and $\{\sqrt{t}\sqrt{\Delta_{D-B,D^*-B^*}(I)}\,e^{-(1-s)tB^*-stD^*}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ is s.i. family on $\mathbb{R}_+\times[0,1]$. Indeed, if, for example, we apply Formula (17) to (A^*,A,C^*,C,I) instead of (A,B,C,D,X), we obtain

$$\mathscr{L}[\mu]\Delta_{A^*,A}(I) - \mathscr{L}[\mu]\Delta_{C^*,C}(I) = \operatorname*{{}^s\!\lim}_{T\to+\infty}\int_{[0,T]}\int_{[0,1]} e^{-(1-s)tA^*-stC^*}(C^*+C-A^*-A)e^{-(1-s)tA-stC}\,t\,ds\,d\mu(t)$$
$$\geqslant 0. \tag{18}$$

This shows that ${}^s\!\lim_{T\to+\infty}$ appearing in (18) is bounded by $\mathscr{L}[\mu]\Delta_{A^*,A}(I)$, and therefore, it is a positively definite operator in $\mathcal{B}(\mathcal{H})$. In other words, $\{\sqrt{t}\sqrt{\Delta_{C^*-A^*,C-A}(I)}\,e^{-(1-s)tA-stC}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ is s.i. on $\mathbb{R}_+\times[0,1]$, and the same is true for the three remaining families. Due to [19] (Lemma 2.1(b)), this proves that Formula (8) holds, while Formula (9) is proved by analogy. Also, by [19] (Lemma 2.1(a1)), this implies that the o.v. integral appearing on the right-hand side of Formula (10) is well defined, while [19] (Lemma 2.1(b)) combined with Formula (17) proves Formula (10).

To prove Formula (11), we conclude directly from Formula (10) that

$$\sqrt{\Delta_{C^*-A^*,C-A}(I)}\left(\mathscr{L}[\mu](A\otimes I+I\otimes B)X - \mathscr{L}[\mu](C\otimes I+I\otimes D)X\right)\sqrt{\Delta_{D-B,D^*-B^*}(I)}$$
$$= \operatorname*{{}^s\!\lim}_{T\to+\infty}\int_{[0,T]}\int_{[0,1]}\sqrt{\Delta_{C^*-A^*,C-A}(I)}\,e^{-(1-s)tA-stC}\big((C-A)X+X(D-B)\big)$$
$$\times e^{-(1-s)tB-stD}\sqrt{\Delta_{D-B,D^*-B^*}(I)}\,t\,ds\,d\mu(t). \tag{19}$$

By taking into account that both families $\{\sqrt{t}\sqrt{\Delta_{C^*-A^*,C-A}(I)}e^{-(1-s)tA-stC}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ and $\{\sqrt{t}\sqrt{\Delta_{D^*-B^*,D-B}(I)}e^{-(1-s)tB-stD}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ are s.i. families on $\mathbb{R}_+\times[0,1]$, we can recognize the right-hand side of Formula (19) as

$$\int_{[0,T]}\int_{[0,1]}\sqrt{\Delta_{C^*-A^*,C-A}(I)}e^{-(1-s)tA-stC}\big((C-A)X+X(D-B)\big)e^{-(1-s)tB-stD}\sqrt{\Delta_{D-B,D^*-B^*}(I)}\,t\,ds\,d\mu(t),$$

according to [19] (Th. 3.1 (c)), which proves Formula (11).

The proof for Formula (12) is quite analogous to that for (11), with the only significant difference being that this time $\{\sqrt{t}e^{-(1-s)tA-stC}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ and $\{\sqrt{t}e^{-(1-s)tB-stD}\}_{0\leqslant t,0\leqslant s\leqslant 1}$ are s.i. families under consideration. □

Note that for $A,B,X\in\mathcal{B}(\mathcal{H})$, the special case $N:=1$ in (5) says $X-e^{-TA}Xe^{-TB} = \int_{[0,T]} e^{-tA}(AX+XB)e^{-tB}dt$ for all $T\geqslant 0$. which immediately implies $e^{-tA}X - Xe^{-tB} = e^{-tA}\left(X - e^{tA}Xe^{-tB}\right) = -\int_{[0,t]} e^{-(t-s)A}(AX-XB)e^{-sB}ds$ for all $t\in\mathbb{R}_+$, which also could be used to prove (13).

3.2. Integral Representation Formula for the Remainder of Taylor Approximation for Laplace Transformers

Recall that for a complex (or finite) Borel measure μ on $\mathbb{R}_+$ and $\overline{\Pi}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \Re z \geqslant 0\}$, its Laplace transform $f := \mathscr{L}[\mu]\colon \overline{\Pi}_+ \to \mathbb{C}\colon z \mapsto \int_{\mathbb{R}_+} e^{-tz}\, d\mu(t)$ is holomorphic in the open right half plane Π^+, and it satisfies $f^{(k)}(z) = (-1)^k \int_{\mathbb{R}_+} t^k e^{-tz}\, d\mu(t)$ for all $n \in \mathbb{N}$. If μ has a finite n-th moment for some $n \in \mathbb{N}$, (i.e., if $\int_{\mathbb{R}_+} t^n d|\mu|(t) < +\infty$), then $(-1)^k \int_{\mathbb{R}_+} t^k e^{-tz}\, d\mu(t)$ is also well defined for $\Re z = 0$ and for any $\mathbb{N}_\circ \ni k \leqslant n$. In this case, $(-1)^k \int_{\mathbb{R}_+} t^k e^{-tC} X e^{-tD}\, d\mu(t)$ is also well defined for all accretive $C, D \in \mathcal{B}(\mathcal{H})$ and for any $X \in \mathcal{B}(\mathcal{H})$.

This leads to the following notation, which will be used in the sequel.

Definition 3. *Let $n \in \mathbb{N}$, $A, B, C, D, X \in \mathcal{B}(\mathcal{H})$ and μ be a Borel measure on $\mathbb{R}_+$. If for $f := \mathscr{L}[\mu]$ operators $f(A) \stackrel{\text{def}}{=} {}^{w^*}\!\!\int_{\mathbb{R}_+} e^{-tA} d\mu(t)$ and $f^{(k)}(B) \stackrel{\text{def}}{=} (-1)^k\, {}^{w^*}\!\!\int_{\mathbb{R}_+} t^k e^{-tB} d\mu(t)$ are well defined for all $\mathbb{N}_\circ \ni k \leqslant n$, then the generalized derivation (g.d.) Taylor polynomial (respectively, remainder) of order n in (a fixed) B (evaluated in a variable A) is defined respectively by*

$$T_n(f;B,A)X \stackrel{\text{def}}{=} \sum_{k=0}^{n} \frac{1}{k!} \Delta_{A,B}^k X f^{(k)}(B), \tag{20}$$

$$R_n(f;B,A)X \stackrel{\text{def}}{=} f(A)X - \sum_{k=0}^{n} \frac{1}{k!} \Delta_{A,B}^k X f^{(k)}(B). \tag{21}$$

If for the generalized derivation transformers $\Delta_{A,B} \stackrel{\text{def}}{=} A \otimes I + I \otimes B$ and $\Delta_{C,D} := C \otimes I + I \otimes D$ the o.v. functions $e^{-tA} X e^{-tB}$ and $\{t^k e^{-tC} X e^{-tD}\}_{k=0}^{n}$ are weakly integrable on $\mathbb{R}_+$ for any $\mathbb{N} \ni k \leqslant n$, then for the Laplace transformer $\mathscr{L}[\mu](\Delta_{A,B})$, its Taylor polynomial (respectively, remainder) transformer of order n in (a fixed) $\Delta_{C,D}$ (evaluated in a variable $\Delta_{A,B}$) is defined by*

$$T_{n;\Delta_{C,D}} f(\Delta_{A,B})X \stackrel{\text{def}}{=} T_n(f;\Delta_{C,D},\Delta_{A,B})X \stackrel{\text{def}}{=} \sum_{k=0}^{n} \frac{1}{k!} (\Delta_{A,B} - \Delta_{C,D})^k f^{(k)}(\Delta_{C,D})X, \tag{22}$$

$$R_{n;\Delta_{C,D}} f(\Delta_{A,B})X \stackrel{\text{def}}{=} R_n(f;\Delta_{C,D},\Delta_{A,B})X \stackrel{\text{def}}{=} f(\Delta_{A,B})X - \sum_{k=0}^{n} \frac{1}{k!} (\Delta_{A,B} - \Delta_{C,D})^k f^{(k)}(\Delta_{C,D})X, \tag{23}$$

where $f^{(k)}(\Delta_{C,D})$ stands for the transformer $(-1)^k \int_{\mathbb{R}_+} t^k e^{-tC} \otimes e^{-tD}\, d\mu(t)$ for any $\mathbb{N}_\circ \ni k \leqslant n$.

Remark 1. *The first thing to note is that for the transformer $R_n(f;B, A)$ defined by (21), the operator $R_n(f;B,A)(I)$ generally does not coincide with $R_p(\varphi;H_\circ,V)$ defined by (1) if $p := n$, $f := \varphi$, $H_\circ := B$ and $V := A - B$, but it does coincide for commuting A and B (i.e. for commuting $H_\circ$ and V). Also, $(\Delta_{A,B} - \Delta_{C,D})^k X = \Delta_{A-C,B-D}^n X = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (A-C)^{n-k-1} X (B-D)^k$. Transformers $f^{(k)}(\Delta_{C,D})$ and $(\Delta_{A,B} - \Delta_{C,D})^k$ commute whenever $\Delta_{A,B}$ and $\Delta_{C,D}$ commute, with the later being satisfied if $AC = CA$ and $BD = DB$.*

The next essential difference between the remainder transformers defined by (1) and (21) is that the Taylor polynomial transformer defined by (22) is an elementary transformer (mapping), which allows the approximation of the generalized functional derivation $f(A)X - Xf(B)$ by the elementary transformer $\sum_{k=1}^{n} \frac{1}{k!} \Delta_{A,B}^k X f^{(k)}(B)$, while the applicability of the remainder in (1) is mainly restricted to functional perturbations $\varphi(H_\circ + V) - \varphi(H_\circ)$.

To consider the estimates for the remainder of the Taylor approximation for Laplace transformers, we need the integral representation formulas presented in the following lemma.

Lemma 2. *If $n \in \mathbb{N}$, μ is a finite Borel measure with finite $(n+1)$-th moment and $A, B, C, D, X \in \mathcal{B}(\mathcal{H})$ are such that A, B, C, D are accretive, $AC = CA$ and $BD = DB$, then for $f := \mathscr{L}[\mu]$ (i.e., $f\colon \overline{\Pi}_+ \to \mathbb{C}\colon z \mapsto \int_{\mathbb{R}_+} e^{-tz}\, d\mu(t)$)*

$$f(\Delta_{A,B})X - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{C,D})(\Delta_{A,B} - \Delta_{C,D})^k X = \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tA} e^{-u(C-A)} \Delta^{n+1}_{C-A,D-B} X e^{-u(D-B)} e^{-tB} u^n \, du\, d\mu(t), \tag{24}$$

$$f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A} - \Delta_{C^*,C})^k(I) = \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tA^*} e^{-u(C-A)^*} \Delta^{n+1}_{C^*-A^*,C-A}(I) e^{-u(C-A)} e^{-tA} u^n \, du\, d\mu(t), \tag{25}$$

$$f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*} - \Delta_{D,D^*})^k(I) = \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tB} e^{-u(D-B)} \Delta^{n+1}_{D-B,D^*-B^*}(I) e^{-u(D-B)^*} e^{-tB^*} u^n \, du\, d\mu(t). \tag{26}$$

Moreover, if $A - C$ and $B^* - D^*$ are additionally $(n+1)$-semiaccretive, then $f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A} - \Delta_{C^*,C})^k(I)$ (respectively, $f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*} - \Delta_{D,D^*})^k(I)$) and $\Delta^{n+1}_{C^*-A^*,C-A}(I)$ (respectively, $\Delta^{n+1}_{D-B,D^*-B^*}(I)$) share the same type of semidefinitness and

$$\Big|f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A} - \Delta_{C^*,C})^k(I)\Big| = \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tA^*} e^{-u(C-A)^*} \big|\Delta^{n+1}_{C^*-A^*,C-A}(I)\big| e^{-u(C-A)} e^{-tA} u^n \, du\, d\mu(t), \tag{27}$$

$$\Big|f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*} - \Delta_{D,D^*})^k(I)\Big| = \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tB} e^{-u(D-B)} \big|\Delta^{n+1}_{D-B,D^*-B^*}(I)\big| e^{-u(D-B)^*} e^{-tB^*} u^n \, du\, d\mu(t). \tag{28}$$

Proof. According to the notation from the Definition 3, it follows that

$$\begin{aligned} f(\Delta_{A,B})X - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\Delta_{C,D})(\Delta_{A,B} - \Delta_{C,D})^k X &= \int_{\mathbb{R}_+} \Big(e^{-tA} X e^{-tB} - \sum_{k=0}^{n} \tfrac{(-t)^k}{k!} e^{-tC} \Delta^k_{A-C,B-D} X e^{-tD}\Big)\, d\mu(t) \\ &= \int_{\mathbb{R}_+} e^{-tA}\Big(X - \sum_{k=0}^{n} \tfrac{t^k}{k!} e^{-t(C-A)} \Delta^k_{C-A,D-B} X e^{-t(D-B)}\Big) e^{-tB} d\mu(t) \\ &= \frac{1}{n!}\int_{\mathbb{R}_+}\int_{[0,t]} e^{-tA} e^{-u(C-A)} \Delta^{n+1}_{C-A,D-B} X e^{-u(D-B)} e^{-tB} u^n \, du\, d\mu(t). \end{aligned} \tag{29}$$

The last equality in (29) is due to the application of Formula (5) to $(C-A, D-B, n)$ instead of $(A, B, n-1)$, which proves Formula (24). The double application of Formula (24), the first time to (A^*, A, C^*, C, I) instead of (A, B, C, D, X), and the second time to (B, D, B^*, D^*, I) instead of (A, B, C, D, X), implies Formulas (25) and (26).

Moreover, if $C - A$ is additionally $(n+1)$-semiaccretive, then Formula (25) shows that $R_n(f; \Delta_{C^*,C}, \Delta_{A^*,A})(I)$ and $\Delta^{n+1}_{C^*-A^*,C-A}(I)$ share the same type of semidefinitness, so due to $|\Delta^{n+1}_{C^*-A^*,C-A}(I)| = \pm\Delta^{n+1}_{C^*-A^*,C-A}(I)$, this immediately implies Formula (27). The repeated use of the same arguments suffices to prove Formula (28) as well. □

Remark 2. *If accretive $A, C \in \mathcal{B}(\mathcal{H})$ commute and $C - A$ is $(n+1)$-accretive, then $R_n(f; \Delta_{C^*,C}, \Delta_{A^*,A})(I) \geqslant 0$ according to Formula (25), so $T_n(f; \Delta_{C^*,C}, \Delta_{A^*,A})(I) \leqslant f(\Delta_{A^*,A})(I)$. In* *particular, for $A := 0$ we have $f(\Delta_{0,0})(I) - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\Delta_{C^*,C})\Delta_{C^*,C}^k(I) = R_n(f;\Delta_{C^*,C},\Delta_{0,0})(I) \geqslant 0$ is C if $(n+1)$-accretive. Similarly, if $C := 0$, and A is $(n+1)$-accretive, then the sign of $R_n(f;\Delta_{0,0},\Delta_{A^*,A})(I) = f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{0,0})\Delta_{A^*,A}^k(I)$ equals to $(-1)^{n+1}$, implying $(-1)^{n+1}R_n(f;\Delta_{0,0},\Delta_{A^*,A})(I) \geqslant 0$.*

4. Main Results

4.1. Q^* Norm Inequalities for the Remainder of the Taylor Approximation of Laplace Transformers

The integral representation Formula (24) allows us to estimate the remainder of the Taylor approximation in (23) for a class of accretive operators.

Theorem 1. *If Ψ is an s.n. function, $n \in \mathbb{N}, \mu$ is a finite Borel measure on $\mathbb{R}_+$ with the finite $(n+1)$th moment, $A, B, C, D, X \in \mathcal{B}(\mathcal{H})$, are such that A, B, C, D are accretive, $C - A$ and $D^* - B^*$ are $(n+1)$-semiaccretive, $AC = CA$, $BD = DB$, and $\Delta_{A-C,B-D}^{n+1} X \in \mathcal{C}_\Psi(\mathcal{H})$, then for $f := \mathscr{L}[\mu]$, the inequality*

$$\begin{aligned}
&\left\| \left|\Delta_{C^*-A^*,C-A}^{n+1}(I)\right|^{\frac{1}{2}} \left(f(\Delta_{A,B})X - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{C,D})(\Delta_{A,B} - \Delta_{C,D})^k X\right) \left|\Delta_{D-B,D^*-B^*}^{n+1}(I)\right|^{\frac{1}{2}} \right\|_\Psi \\
&\leqslant \left\| \left| f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A} - \Delta_{C^*,C})^k(I) \right|^{\frac{1}{2}} \sum_{k=0}^{n+1} \binom{n+1}{k} (A-C)^{n+1-k} X (B-D)^k \right. \\
&\quad \left. \times \left| f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*} - \Delta_{D,D^*})^k(I) \right|^{\frac{1}{2}} \right\|_\Psi, \qquad (30)
\end{aligned}$$

holds under any of the additional conditions:
(a) $\Psi := \ell^1$*, i.e., $\|\cdot\|_\Psi$ is the trace norm $\|\cdot\|_1$;*
(b) $\Psi := \Phi^{(p)^*}$ *for some $p \geqslant 2$ and at least one of the pairs (A, C^*) and (B, D^*) consists of normal operators;*
(c) *Both pairs (A, C^*) and (B, D^*) consist of normal operators.*

Proof. If $D^* - B^*$ is $(n+1)$-accretive, then the proof for Ineq. (30) relies on the following calculations:

$$\begin{aligned}
&\left\| \left|\Delta_{C^*-A^*,C-A}^{n+1}(I)\right|^{1/2} \left(f(\Delta_{A,B})X - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{C,D})(\Delta_{A,B} - \Delta_{C,D})^k X\right) \left|\Delta_{D-B,D^*-B^*}^{n+1}(I)\right|^{1/2} \right\|_\Psi \\
&= \frac{1}{n!} \left\| \left|\Delta_{C^*-A^*,C-A}^{n+1}(I)\right|^{1/2} \int_{\mathbb{R}_+} \int_0^t e^{-tA} e^{-u(C-A)} \Delta_{C-A,D-B}^{n+1} X e^{-u(D-B)} e^{-tB} u^n \, du \, d\mu(t) \left|\Delta_{D-B,D^*-B^*}^{n+1}(I)\right|^{1/2} \right\|_\Psi \qquad (31) \\
&\leqslant \frac{1}{n!} \left\| \left(\int_{\mathbb{R}_+} \int_0^t e^{-tA^*} e^{-u(C-A)^*} \left|\Delta_{C^*-A^*,C-A}^{n+1}(I)\right| e^{-tA} e^{-u(C-A)} u^n du d\mu(t) \right)^{1/2} \Delta_{C-A,D-B}^{n+1} X \right. \\
&\quad \left. \times \left(\int_{\mathbb{R}_+} \int_0^t e^{-u(D-B)} e^{-tB} \left|\Delta_{D-B,D^*-B^*}^{n+1}(I)\right| e^{-tB^*} e^{-u(D-B)^*} u^n du d\mu(t) \right)^{1/2} \right\|_\Psi \qquad (32) \\
&= \frac{1}{n!} \left\| \left| \int_{\mathbb{R}_+} \int_0^t e^{-tA^*} e^{-u(C-A)^*} \Delta_{C^*-A^*,C-A}^{n+1}(I) e^{-tA} e^{-u(C-A)} u^n du d\mu(t) \right|^{1/2} \Delta_{C-A,D-B}^{n+1} X \right. \\
&\quad \left. \times \left| \int_{\mathbb{R}_+} \int_0^t e^{-u(D-B)} e^{-tB} \Delta_{D-B,D^*-B^*}^{n+1}(I) e^{-tB^*} e^{-u(D-B)^*} u^n du d\mu(t) \right|^{1/2} \right\|_\Psi \qquad (33) \\
&= \left\| \left| f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A} - \Delta_{C^*,C})^k(I) \right|^{1/2} \Delta_{A-C,B-D}^{n+1} X \right. \\
&\quad \left. \times \left| f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*} - \Delta_{D,D^*})^k(I) \right|^{1/2} \right\|_\Psi. \qquad (34)
\end{aligned}$$

Here, the equality in (31) is based on the integral representation identities (29), while Ineq. (32) is based on the application of the Cauchy–Schwarz norm inequalities to s.i. families $\{\,|\Delta^{n+1}_{C^*-A^*,C-A}(I)|^{1/2}e^{-tA}u^n e^{-u(C-A)}\,\}_{0\leqslant u\leqslant t}$ and $\{|\Delta^{n+1}_{D-B,D^*-B^*}(I)|^{1/2}e^{-tB^*}\;u^n e^{-u(D-B)^*}\}_{0\leqslant u\leqslant t}$. Specifically, in case (a), we apply Ineq. (30) in [19] (Th. 3.1 (c)); in case (b), we apply the first inequality in (32) in [19] (Th. 3.1 (d)); and in case (c), we apply Ineq. (32) in [19] (Th. 3.1 (e)). Moreover, Equation (33) is based on Formulas (27) and (28), while the equality in (34) follows by the integral representation Formulas (25) and (26). □

Corollary 1. *If μ is a Borel probability measure on $\mathbb{R}_+$ with a finite $(n+1)$th moment, $A,B,X\in\mathcal{B}(\mathcal{H})$ are such that A,B^* are $(n+1)$-accretive and $\Delta^{n+1}_{A,B}X\in\mathcal{C}_\Psi(\mathcal{H})$ for an s.n. function Ψ, then for $f:=\mathscr{L}[\mu]$, we have $f(0)=1$, and*

$$\begin{aligned}&\Big\|\sqrt{\Delta^{n+1}_{A^*,A}(I)}\Big(f(\Delta_{A,B})X-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(0)\Delta^k_{A,B}X\Big)\sqrt{\Delta^{n+1}_{B,B^*}(I)}\Big\|_\Psi\\&\leqslant\Big\|\Big((-1)^{n+1}\Big(f(\Delta_{A^*,A})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(0)\Delta^k_{A^*,A}(I)\Big)\Big)^{\frac12}\Delta^{n+1}_{A,B}X\\&\quad\times\Big((-1)^{n+1}\Big(f(\Delta_{B,B^*})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(0)\Delta^k_{B,B^*}(I)\Big)\Big)^{\frac12}\Big\|_\Psi,\end{aligned}\tag{35}$$

$$\begin{aligned}&\Big\|\sqrt{\Delta^{n+1}_{A^*,A}(I)}\Big(f(0)X-\sum_{k=0}^{n}\tfrac{(-1)^k}{k!}f^{(k)}(\Delta_{A,B})\Delta^k_{A,B}X\Big)\sqrt{\Delta^{n+1}_{B,B^*}(I)}\Big\|_\Psi\\&\leqslant\Big\|\Big(f(0)I-\sum_{k=0}^{n}\tfrac{(-1)^k}{k!}f^{(k)}(\Delta_{A^*,A})\Delta^k_{A^*,A}(I)\Big)^{\frac12}\Delta^{n+1}_{A,B}X\Big(f(0)I-\sum_{k=0}^{n}\tfrac{(-1)^k}{k!}f^{(k)}(\Delta_{B,B^*})\Delta^k_{B,B^*}(I)\Big)^{\frac12}\Big\|_\Psi\end{aligned}\tag{36}$$

hold under any of the following conditions:
(a) $\Psi:=\ell^1$;
(b) $\Psi:=\Phi^{(p)^*}$ *for some $p\geqslant2$ and at least one of operators A or B^* is normal;*
(c) *Both A and B^* are normal operators.*

If A,B^* are additionally $(n+1)$-hyperaccretive, then the expression on the right-hand side in (36) further estimated from above by $f(0)\big\|\sum_{k=0}^{n+1}\binom{n+1}{k}A^{n-k+1}XB^k\big\|_\Psi$.

Proof. Formula (35) is just a special case $C:=D:=0$ of Ineq. (30), combined with Formula (27). Similarly, Formula (36) is also the special case of (30), applied to $(0,0,A,B)$ instead of (A,B,C,D), this time combined with Formula (28). Finally, if A,B^* are $(n+1)$-hyperaccretive, then for $f^{(k)}(z)=(-1)^k\int_{\mathbb{R}_+}t^n\,d\mu(t)$, we have $(-1)^k f^{(k)}(\Delta_{A^*,A})\Delta^k_{A^*,A}(I)=\int_{\mathbb{R}_+}e^{-tA^*}\Delta^k_{A^*,A}(I)e^{-tA}t^n\,d\mu(t)\geqslant0$ for any $\mathbb{N}_\circ\ni k\leqslant n$, so $\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(\Delta_{A^*,A})\Delta^k_{A^*,A}(I)\geqslant0$, as well as $0\leqslant f(0)I-\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(\Delta_{A^*,A})\Delta^k_{A^*,A}(I)\leqslant f(0)I$, according to Remark 2. Similarly $0\leqslant f(0)I-\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(\Delta_{B,B^*})\Delta^k_{B,B^*}(I)\leqslant f(0)I$, so the additional estimate deduces based on the monotonicity property (2) (i.e., (1) in [14]). □

Corollary 2. *If $A,D,X\in\mathcal{B}(\mathcal{H})$ are such that A,D^* are $(n+1)$-accretive for some $n\in\mathbb{N}$ and $\Delta^{n+1}_{A,-D}X\in\mathcal{C}_\Psi(\mathcal{H})$ for some s.n. function Ψ, then*

$$\begin{aligned}&\Big\|\big(\Delta^{n+1}_{A^*,A}(I)\big)^{\frac12}\Big(f(A)X-\sum_{k=0}^{n}\frac{1}{k!}\Delta^k_{A,-D}Xf^{(k)}(D)\Big)\big(\Delta^{n+1}_{D,D^*}(I)\big)^{\frac12}\Big\|_\Psi\\&\leqslant\Big\|\Big((-1)^{n+1}\Big(f(\Delta_{A^*,A})(I)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(0)\Delta^k_{A^*,A}(I)\Big)\Big)^{\frac12}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}A^{n+1-k}XD^k\\&\quad\times\Big(f(0)(I)-\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(\Delta_{D,D^*})\big(\Delta^k_{D,D^*}(I)\big)\Big)^{\frac12}\Big\|_\Psi\end{aligned}\tag{37}$$

holds under any of the following conditions:
(a) $\Psi:=\ell^1$, *i.e., $\|\cdot\|_\Psi$ is the trace norm $\|\cdot\|_1$;*

(b) $\Psi := \Phi^{(p)^*}$ *for some* $p \geqslant 2$ *and at least one of operators* A *or* B^* *is normal;*
(c) *Both* A *and* D *are normal operators.*

Proof. We just have to recognize that in the special case $B := C := 0$ of Ineq. (30), we have $f^{(k)}(\Delta_{A,B})X = f^{(k)}(A)X$ and $f^{(k)}(\Delta_{C,D})X = Xf^{(k)}(D)$ for all $\mathbb{N}_\circ \ni k \leqslant n$, so the left-hand sides of (30) and (37) in this case coincide. That the same is true for their right-hand sides, we have seen already in the proof of Corollary 1. □

4.2. Q Norm Inequalities for the Remainder of the Taylor Approximation of Laplace Transformers

It is not surprising that norm inequalities for Q and Q* norms have a different (algebraic) forms. Probably the best known examples for this noncommutative phenomenon is a pair of Clarkson–McCarthy inequalities for Shatten–von Neumann norms, saying that

$$||A+B||_p^p + ||A-B||_p^p \leqslant 2^{p-1}\big(||A||_p^p + ||B||_p^p\big) \qquad \text{for } 2 \leqslant p < +\infty,$$
$$||A+B||_p^{\frac{p}{p-1}} + ||A-B||_p^{\frac{p}{p-1}} \leqslant 2\big(||A||_p^{\frac{p}{p-1}} + ||B||_p^{\frac{p}{p-1}}\big)^{\frac{1}{p-1}} \qquad \text{for } 1 \leqslant p \leqslant 2;$$

see [28] (Th. 1.21). Another examples is provided by the Cauchy–Schwarz norm inequalities in [18,19], which should be used to derive the next Q norm inequalities for the remainder of the Taylor approximation of Laplace transformers.

Theorem 2. *Let* Ψ *be an s.n. function,* $n \in \mathbb{N}$, μ *be a finite Borel measure on* $\mathbb{R}_+$ *with the finite* $(n+1)$*th moment,* $A, B, C, D, X \in \mathcal{B}(\mathcal{H})$ *be such that* A, B, C, D *are accretive,* $AC = CA$, $BD = DB$, $\Delta^{n+1}_{A-C,B-D}X \in \mathcal{C}_\Psi(\mathcal{H})$, $\Delta^{n+1}_{C-A,C^*-A^*}(I)$ *be invertible, and* $f := \mathscr{L}[\mu]$.
If $C^* - A^*$ *and* $D^* - B^*$ *are* $(n+1)$*-accretive, then the inequality*

$$\begin{aligned}
&\Big\|\Big(f(\Delta_{A,B})X - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C,D})(\Delta_{A,B}-\Delta_{C,D})^k X\Big)\big(\Delta^{n+1}_{D-B,D^*-B^*}(I)\big)^{\frac12}\Big\|_\Psi \\
&\leqslant \Big\|f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A}-\Delta_{C^*,C})^k(I)\Big\|^{\frac12} \\
&\times\Big\|\big(\Delta^{n+1}_{C-A,C^*-A^*}(I)\big)^{-\frac12}\Delta^{n+1}_{A-C,B-D}X\Big(f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*}-\Delta_{D,D^*})^k(I)\Big)^{\frac12}\Big\|_\Psi \qquad (38)
\end{aligned}$$

holds under any of the additional conditions:
(a) $\Psi := \Phi^{(p)}$ *for some* $p \geqslant 2$ *and* (B, D) *consists of normal operators,*
(b) $\Psi := \ell^2$, *i.e.,* $||\cdot||_\Psi$ *is the Hilbert-Schmidt norm* $||\cdot||_2$.
Alternatively, if $C - A$ *and* $D - B$ *are* $(n+1)$*-accretive, then the inequality*

$$\begin{aligned}
&\Big\|\big(\Delta^{n+1}_{C^*-A^*,C-A}(I)\big)^{\frac12}\Big(f(\Delta_{A,B})X - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C,D})(\Delta_{A,B}-\Delta_{C,D})^k X\Big)\Big\|_\Psi \\
&\leqslant \Big\|f(\Delta_{B,B^*})(I) - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*}-\Delta_{D,D^*})^k(I)\Big\|^{\frac12} \\
&\times\Big\|\Big(f(\Delta_{A^*,A})(I) - \sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A}-\Delta_{C^*,C})^k(I)\Big)^{\frac12}\Delta^{n+1}_{A-C,B-D}X\big(\Delta^{n+1}_{D^*-B^*,D-B}(I)\big)^{-\frac12}\Big\|_\Psi \qquad (39)
\end{aligned}$$

holds under any of the additional conditions:
(c) $\Psi := \Phi^{(p)}$ *for some* $p \geqslant 2$ *and* (A, C) *consists of normal operators;*
(d) $\Psi := \ell^2$.

Proof. The next equality in (40) is based again on the integral representation identities (29),

$$\begin{aligned}
&\left\|\left(f(\Delta_{A,B})X-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C,D})(\Delta_{A,B}-\Delta_{C,D})^{k}X\right)(\Delta^{n+1}_{D-B,D^*-B^*}(I))^{\frac{1}{2}}\right\|_{\Psi}\\
&=\frac{1}{n!}\left\|\int_{\mathbb{R}_+}\int_0^t e^{-tA}e^{-u(C-A)}\Delta^{n+1}_{C-A,D-B}Xe^{-u(D-B)}e^{-tB}u^n du\, d\mu(t)\,(\Delta^{n+1}_{D-B,D^*-B^*}(I))^{\frac{1}{2}}\right\|_{\Psi} \qquad (40)\\
&=\frac{1}{n!}\left\|\int_{\mathbb{R}_+}\int_0^t e^{-tA}e^{-u(C-A)}(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{1/2}(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{-\frac{1}{2}}\Delta^{n+1}_{C-A,D-B}X\right.\\
&\qquad\left.\times\, e^{-u(D-B)}e^{-tB}(\Delta^{n+1}_{D-B,D^*-B^*}(I))^{\frac{1}{2}}u^n du\, d\mu(t)\right\|_{\Psi}\\
&\leqslant\frac{1}{n!}\left\|\int_{\mathbb{R}_+}\int_0^t e^{-tA}e^{-u(C-A)}\Delta^{N+1}_{C-A,C^*-A^*}(I)e^{-tA^*}e^{-u(C-A)}u^n du\, d\mu(t)\right\|^{\frac{1}{2}}\\
&\times\left\|(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{-\frac{1}{2}}\Delta^{n+1}_{C-A,D-B}X\left(\int_{\mathbb{R}_+}\int_0^t e^{-u(D-B)}e^{-tB}(\Delta^{n+1}_{D-B,D^*-B^*}(I))e^{-tB^*}e^{-u(D-B)^*}u^n du d\mu(t)\right)^{\frac{1}{2}}\right\|_{\Psi} \qquad (41)\\
&=\left\|f(\Delta_{A^*,A})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A}-\Delta_{C^*,C})^{k}(I)\right\|^{\frac{1}{2}}\\
&\times\left\|(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{-\frac{1}{2}}\Delta^{n+1}_{A-C,B-D}X\left(f(\Delta_{B,B^*})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*}-\Delta_{D,D^*})^{k}(I)\right)^{\frac{1}{2}}\right\|_{\Psi}. \qquad (42)
\end{aligned}$$

while Ineq. (41) is based on the application of the Cauchy–Schwarz norm inequalities to s.i. families $\{(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{1/2}e^{-tA^*}e^{-u(C-A)^*}u^n\}_{0\leqslant u\leqslant t}$, $\{(\Delta^{n+1}_{D-B,D^*-B^*}(I))^{1/2}e^{-tB^*}e^{-u(D-B)^*}u^n\}_{0\leqslant u\leqslant t}$, also applied to $(\Delta^{n+1}_{C-A,C^*-A^*}(I))^{-\frac{1}{2}}\Delta^{n+1}_{A-C,B-D}X$ instead of X. To specify, in the case (a), we apply Ineq. (29) in [19] (Th. 3.1 (a)), while in the case (b), we apply the inequality in (29) related to [19] (Th. 3.1 (b)). Finally, the equality in (42) is derived by the double application of the identities (29), first to (A, C, A^*, C^*, I) instead of (A, B, C, D, X), and later to (B, D, B^*, D^*, I) instead of (A, B, C, D, X), which proves Ineq. (38).

The proof for Ineq. (39) follows from

$$\begin{aligned}
&\left\|(\Delta^{n+1}_{C^*-A^*,C-A}(I))^{\frac{1}{2}}\left(f(\Delta_{A,B})X-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C,D})(\Delta_{A,B}-\Delta_{C,D})^{k}X\right)\right\|_{\Psi}\\
&=\left\|\left(f(\Delta_{B^*,A^*})X^*-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D^*,C^*})(\Delta_{B^*,A^*}-\Delta_{D^*,C^*})^{k}X^*\right)(\Delta^{n+1}_{C^*-A^*,C-A}(I))^{\frac{1}{2}}\right\|_{\Psi} \qquad (43)\\
&\leqslant\left\|f(\Delta_{B,B^*})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*}-\Delta_{D,D^*})^{k}(I)\right\|^{\frac{1}{2}}\\
&\times\left\|(\Delta^{n+1}_{D^*-B^*,D-B}(I))^{-\frac{1}{2}}\Delta^{n+1}_{B^*-A^*,D^*-C^*}X^*\left(f(\Delta_{A^*,A})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A}-\Delta_{C^*,C})^{k}(I)\right)^{\frac{1}{2}}\right\|_{\Psi} \qquad (44)\\
&=\left\|f(\Delta_{B,B^*})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{D,D^*})(\Delta_{B,B^*}-\Delta_{D,D^*})^{k}(I)\right\|^{\frac{1}{2}}\\
&\times\left\|\left(f(\Delta_{A^*,A})(I)-\sum_{k=0}^{n}\tfrac{1}{k!}f^{(k)}(\Delta_{C^*,C})(\Delta_{A^*,A}-\Delta_{C^*,C})^{k}(I)\right)^{\frac{1}{2}}\Delta^{n+1}_{A-C,B-D}X(\Delta^{n+1}_{D^*-B^*,D-B}(I))^{-\frac{1}{2}}\right\|_{\Psi}, \qquad (45)
\end{aligned}$$

where the equalities in (43) and (45) are due to the norm properties for adjoint operators, while the inequality in (44) is obtained by applying Ineq. (38) to $(B^*, A^*, D^*, C^*, X^*)$ instead of (A, B, C, D, X). □

Remark 3. *The invertibility condition required in Theorem 2 is satisfied if $\Delta^{n+1}_{C-A,C^*-A^*}(I) \gg 0$, i.e., if $\Delta^{n+1}_{C-A,C^*-A^*}(I)$ is uniformly $(n+1)$-accretive. As $\Delta^{n+1}_{C-A,C^*-A^*}(I) \geqslant 0$, then it is right or left invertible if and only if it is invertible.*

5. Conclusions

Concluding this paper, we briefly outline the prospects for future developments in this field of investigation, as the problems related to Laplace transformers open many interesting and promising questions. So, the manuscript containing the Schatten–von Neumann norm inequalities for perturbations of Laplace transformers of accretive derivations, which complements the results published in [16], is expected to be submitted for publication very soon.

The problems considered in this paper arise from the question of approximating the function derivation transformer $X \mapsto f(A)X - Xf(D)$ by elementary (mappings) transformers, in our case by $\sum_{k=1}^{n} \frac{1}{k!} \Delta_{A,-D}^{k} X f^{(k)}(D)$, which involves higher-order generalized derivations $\Delta_{A,-D}^{k}\colon X \mapsto \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-1} X D^i$. Thus, Ineq. (37) in Corollary 2 gives the weighted estimate, which provides one of the answers to the raised approximation question. Work involving estimates for some other types of approximation by elementary transformers is in preparation and should contribute to the development of this (transformers) type of noncommutative (generalization of the standard) calculus.

Norm inequalities for Taylor reminders for transformers related to analytic functions with non-negative Taylor coefficients have already been obtained, and the corresponding manuscript is in preparation, while the analysis of Taylor reminders for transformers related to other classes of holomorphic functions are also advancing.

Probably the most important step to develop the applications of derivations and/or Taylor reminders for Laplace transformers related to (hyper) accretive operators is the development of the analog of the (higher-order) spectral shift function in the adequate context.

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