Determination of particular double starlike trees by the Laplacian spectrum

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Abstract

A double starlike tree is a tree in which exactly two vertices have degree greater than two. In this study we consider double starlike trees obtained by attaching $p-2$ (for $p \geq 3$) pendant vertices at an internal vertex and $q-2$ ($q \geq 3$) pendant vertices at a different internal vertex of a fixed path P. We denote this tree by $T \cong D(a,b,c,p,q)$, where a,b,c stand for the numbers of vertices in segments of P obtained by deleting vertices of degree p and q. It is known that, depending on parameters, T may or may not be determined by its Laplacian spectrum. In the latter case we provide the structure of a putative tree with the same Laplacian spectrum. This result implies the known result stating that $D(1,b,1,p,q)$ is determined by the Laplacian spectrum and two new results stating the same for $D(1,b,2,p,p)$ and $D(2,b,2,p,p)$.

Keywords: Double starlike tree, Laplacian matrix, Vertex degree, Path
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1. Introduction

The *Laplacian matrix* of a graph $G = G(V,E)$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the standard adjacency matrix of G. The *Laplacian eigenvalues* of G are identified to be the eigenvalues of $L(G)$, and they form the *Laplacian spectrum* of G.

We say that two graphs are *cospectral* (resp. *Laplacian cospectral*) if they are not isomorphic but share the same spectrum of the adjacency (Laplacian) matrix. We say that a graph is *determined by the Laplacian spectrum* if there is no graph which is Laplacian cospectral with it.

A *starlike tree* is a tree with exactly one vertex of degree greater than two, and we know from [9] that every such a tree is determined by its Laplacian spectrum. The next natural step is to consider *double starlike trees*, i.e. those with exactly two vertices of degree greater than two. We denote a particular class of double starlike trees obtained by attaching $p-2$ pendant vertices at one and $q-2$ pendant vertices at one other vertex of a fixed path by $D(a,b,c,p,q)$, where a,b,c denote the numbers of vertices in three segments of the path, along with $p \geq q \geq 3, a,c \geq 1, b \geq 0$, see Fig. 1(a). The segments do not include vertices of degree p and q, and can be seen as induced paths obtained by deleting these two vertices. Accordingly, $D(a,b,c,p,q)$ has $a+b+c+p+q-2$ vertices. We remark that double starlike trees considered in this paper belong to the class of caterpillars, i.e. trees in which a deletion of all pendant vertices results in a path [10, 11].

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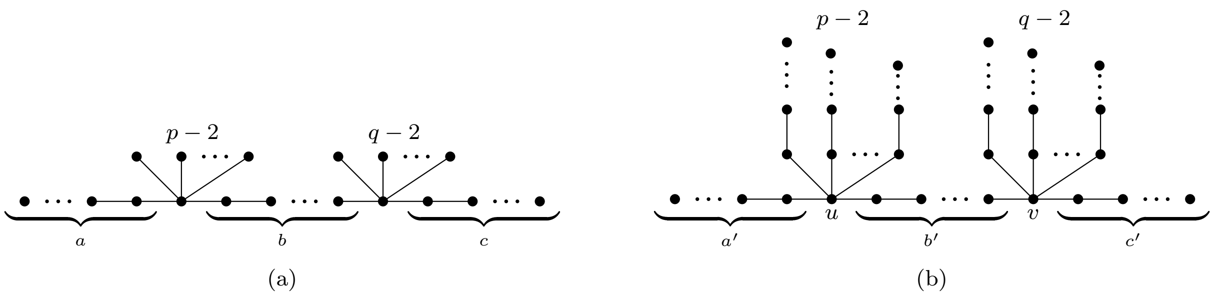

Figure 1: (a) A double starlike tree $D(a, b, c, p, q)$. (b) A tree $M(a', b', c', p, q)$.

We know from [11] that $D(a, b, c, 3, 3)$ is not determined by the Laplacian spectrum if and only if $(a, b, c) = (3, 0, 4)$ or $(b, c) = (2, a + 1)$ with $a \geq 2$. We also know from [6, 8] that $D(1, b, 1, p, q)$ is determined by the Laplacian spectrum. Therefore, depending on parameters, a tree under consideration may or may not be determined by the Laplacian spectrum. This motivates us to consider the structure of a putative graph with the same Laplacian spectrum. We define $M(a', b', c', p, q)$ to be the tree illustrated in Fig. 1(b) where, as before, a', b', c' denote the numbers of vertices in three segments of a fixed path, and now $p-2$ and $q-2$ denote the numbers of paths attached at distinct internal vertices u and v (of the same path). As before, $p \geq q \geq 3, a', c' \geq 1, b' \geq 0$, and u, v are not included in any of three segments. Without loss of generality, we also assume that the length of a hanging path having u (resp. v) as an endvertex does not exceed $a'(c')$.

Throughout the paper, when we deal with the length of a hanging path attached at a fixed vertex, say u, we always assume that u is its endvertex. Accordingly, a pendant vertex attached at u can be seen as a path of length 1.

Our main contribution reads as follows.

Theorem 1.1. *If a double starlike tree $D(a, b, c, p, q)$ is Laplacian cospectral with a graph G, then $G \cong M(a', b', c', p, q)$ and at least one of the inequalities*

$$a' \geq a,\ b' \leq b,\ c' \geq c \tag{1}$$

fails to hold.

As an application, we give a new proof of the aforementioned result stating that $D(1, b, 1, p, q)$ is determined by the Laplacian spectrum, and we also prove that the same holds for $D(1, b, 2, p, p)$ and $D(2, b, 2, p, p)$.

Section 2 is preparatory. The proof of Theorem 1.1 is given in Section 3. The mentioned consequences are proved in Section 4.

2. Preliminaries

We denote by d_i the degree of a vertex $v_i \in V(G)$, and by $\deg(G) = (d_1, d_2, \ldots, d_n)$ the collection of vertex degrees in G arranged in the form of a non-increasing sequence. We also write d_{uv} for the degree of an edge uv, where $d_{uv} = d_u + d_v - 2$. For the graphs G and H, $G \cup H$ denotes their disjoint union, and kG denotes the disjoint union of k copies of G. The *line graph* $\mathrm{Line}(G)$ of G is the graph whose vertices are the edges of G, with two vertices in $\mathrm{Line}(G)$ adjacent if and only if the corresponding edges are adjacent in G. It is well-known that the Laplacian eigenvalues of a graph are real, and we denote them by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. We also use λ_1, λ_2 and κ_1 to denote the largest eigenvalue of $A(G)$, the second largest eigenvalue of the same matrix and the largest eigenvalue of $D(G) + A(G)$ (the *signless Laplacian matrix*), respectively. For basic notions on graphs not given here, we refer the reader to [3, 10]. We list some known results.

Theorem 2.1 ([3]). *The following invariants are determined by the Laplacian spectrum of a graph G:*

(i) Connectedness (i.e. whether G is connected or not);
(ii) The number of vertices;
(iii) The number of edges;
(iv) The sum of squares of degrees of vertices.

Theorem 2.2 ([5] or [10, p. 159]). *For a graph G with at least one edge,*

$$\mu_1(G) \leq \max\left\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} \ : \ uv \in E(G)\right\}, \tag{2}$$

where m_u is the average vertex degree in the neighbourhood of u. Equality holds if and only if G is bipartite and regular.

Theorem 2.3 ([2]). *If a graph G is not isomorphic to $K_i \cup (n-i)K_1$, then $\mu_i(G) \geq d_i - i + 2$ holds for $1 \leq i \leq n-1$.*

A *subdivision* of an edge e in a graph G consists of replacing e with a path of length 2.

Theorem 2.4. *Let H be obtained from a connected graph G by subdividing its edge e. If G and H are bipartite, then we have the following:*

$$\mu_1(H) \begin{cases} > \mu_1(G), & \text{if } e \text{ is on a hanging path,} \\ < \mu_1(G), & \text{otherwise.} \end{cases}$$

Proof. If G and H are bipartite, then from [3, Theorem 7.8.4], we have $\mu_1(G) = \kappa_1(G)$ and $\mu_1(H) = \kappa_1(H)$. Observe also that bipartitenness excludes the possibility that G is a cycle. The desired result follows from [4, Theorem 2.9] stating that $\kappa_1(H) > \kappa_1(G)$ if e is on a hanging path and $\kappa_1(H) < \kappa_1(G)$ if e is not on a hanging path and G is not a cycle. □

Theorem 2.5 ([7]). *Let $G - u$ be the subgraph of a graph G with n vertices obtained by deleting a vertex u. Then $\mu_i(G) \geq \mu_i(G-u) \geq \mu_{i+1}(G) - 1$ holds for $1 \leq i \leq n-1$.*

Theorem 2.6 (cf. [3, pp. 216–217]). *If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the Laplacian eigenvalues of a tree T then $\mu_i - 2$ $(1 \leq i \leq n-1)$ are the eigenvalues (of the adjacency matrix) of* $\mathrm{Line}(T)$.

We proceed with the characteristic polynomial of $A(G)$ and a particular coefficient of the characteristic polynomial of $L(T)$.

Theorem 2.7 ([3, Theorem 2.2.4]). *Let G be the graph obtained by inserting an edge between a vertex u of a graph G_1 and a vertex v of a graph G_2. Then*

$$\Phi_G(x) = \Phi_{G_1}(x)\Phi_{G_2}(x) - \Phi_{G_1-u}(x)\Phi_{G_2-v}(x),$$

where Φ_G stands for the characteristic polynomial of the adjacency matrix of G.

Theorem 2.8 ([1, Theorem 3.1]). *If* $\sum_{i=0}^{n}(-1)^{n-k}c_i(T)x^i$ *is the characteristic polynomial of the Laplacian matrix of a tree* T*, then*

$$
\begin{aligned}
c_{n-4}(T) = {} & (n-1)\Big(\frac{2}{3}n^3 - 4n^2 + \frac{29}{2}n - \frac{266}{3}\Big) - \frac{9}{4}M_1^4(T) + \frac{17}{8}M_1^2(T) - \Big(n^2 - \frac{7}{2}n - 80\Big)M_1(T) \\
& - 40M_2(T) + \frac{39}{2}EM_1(T) + \Big(\frac{2}{3}n - \frac{103}{6}\Big)F(T) - 16\sum_{u,v\in V(T)}\binom{d_u}{2}\binom{d_v}{2},
\end{aligned}
$$

where $M_1(T) = \sum_{i=1}^{n} d_i^2$, $EM_1(T) = \sum_{uv\in E(T)} d_{uv}^2$, $M_2(T) = \sum_{uv\in E(T)} d_u d_v$, $F(T) = \sum_{i=1}^{n} d_i^3$.

An experienced reader will recognize that $M_1(T), M_2(T), F(T)$ and $EM_1(T)$ denote the first Zagreb index, the second Zagreb index, the forgotten topological index and the edge-counterpart to the first Zagreb index, respectively.

3. The proof

The proof of Theorem 1.1 relies on the following two lemmas. For the proof of the first one we use the idea of [8]. However, in a particular point of the proof we show that $\deg(T) = \deg(G)$ holds whenever $q \leq 6$, and in this way we avoid consideration of several particular cases, which reduces the entire proof. For the sake of completeness we give full details.

Lemma 3.1. *If* $T \cong D(a, b, c, p, q)$ *and* G *are Laplacian cospectral, then* G *is a tree and* $\deg(T) = \deg(G)$.

Proof. From Theorem 2.1(i) and (iii), G is a tree. Let $\deg(G) = (d_1, d_2, \dots, d_n)$.

First, we fix some upper bounds for d_1, d_2 and d_3. From Theorem 2.3 we have $d_1 \leq \mu_1 - 1$. Applying Theorem 2.2 to T, we obtain $\mu_1 < p + 2$. Indeed, by the assumption we have $p \geq q$, and then the right hand side of (2) does not exceed $p + 2$, and it attains this value for $p = q$ and $b = 0$, but since T is non-regular, μ_1 is strictly less than $p + 2$. The obtained inequalities give $d_1 \leq p$.

Concerning d_2 we get the following inequalities: $d_2 \leq \mu_2(G) \leq \mu_1(G - u) + 1 < q + 3$, where the first inequality follows from Theorem 2.3, the second one follows from Theorem 2.5 and the third one follows from Theorem 2.2. Hence, $d_2 \leq q + 2$.

Finally, considering the matrix obtained from $L(G)$ by deleting rows and columns that correspond to vertices u, v and using the eigenvalue interlacing, we obtain $\mu_3 < 4$. Then, Theorem 2.3 gives $d_3 \leq 4$.

We proceed with some algebraic calculus. By computing the number of edges in G, the sum of squares of its vertex degrees and using Theorem 2.1(iii)–(iv), we arrive at

$$
\sum_{i=1}^{n} d_i = 2(n-1) \quad \text{and} \quad \sum_{i=1}^{n} d_i^2 = p^2 + q^2 - 3(p+q) + 2(2n-1), \tag{3}
$$

where the second equality follows since T has $p + q - 2$ vertices of degree 1, $a + b + c - 2 = n - p - q$ vertices of degree 2, one vertex of degree q and one vertex of degree p.

We know from Theorem 2.6 and [3, p. 57] that the number of triangles in $\mathrm{Line}(T)$ is equal to the number of triangles in $\mathrm{Line}(G)$. This yields

$$
\sum_{i=1}^{n}\binom{d_i}{3} = \frac{1}{6}\sum_{i=1}^{n} d_i(d_i^2 - 3d_i + 2) = \binom{p}{3} + \binom{q}{3}. \tag{4}
$$

If u and v are fixed vertices of degree d_1 and d_2 in G, respectively, then we denote by t_i the number of vertices that are distinct from u and v, and have degree i for $i \in \{3,4\}$. The equality (4) can be rewritten as

$$\binom{d_1}{3} + \binom{d_2}{3} + 4t_4 + t_3 = \binom{p}{3} + \binom{q}{3}. \tag{5}$$

We will return to this form soon. At this moment, by using (3) and (4), we compute

$$\sum_{i=1}^{n}(d_i-1)(d_i-2)(d_i-3) = 6\sum_{i=1}^{n}\binom{d_i}{3} - 3\sum_{i=1}^{n}(d_i^2-3d_i+2) = 6\left(\binom{p-1}{3}+\binom{q-1}{3}\right).$$

Since $(d_i-1)(d_i-2)(d_i-3) = 0$ holds whenever $d_i \leq 3$, the previous equality reduces to

$$\sum_{i=1}^{2}(d_i-1)(d_i-2)(d_i-3) + 6t_4 = 6\left(\binom{p-1}{3}+\binom{q-1}{3}\right),$$

which gives

$$t_4 = \binom{p-1}{3} + \binom{q-1}{3} - \binom{d_1-1}{3} - \binom{d_2-1}{3}. \tag{6}$$

In what follows we prove that $d_1 = p, d_2 = q, t_3 = 0$ and $t_4 = 0$. There are two cases depending on d_1.

Case 1: $d_1 < p$. (This possibility will be eliminated by way of contradiction.) Using (4) we obtain

$$\frac{p-1}{6}\sum_{i=1}^{n}(d_i^2-3d_i+2) \geq \binom{p}{3} + \binom{q}{3}.$$

Using (3) we compute the left hand side to obtain

$$\frac{p-1}{6}\big(p^2+q^2-3(p+q)+4\big) \geq \binom{p}{3} + \binom{q}{3}.$$

After simplifying, the previous inequality becomes

$$p^2 - (q(q-3)+5)p \leq -q^3 + 2q^2 + q - 4.$$

Now, it is a matter of routine to confirm that this inequality does not hold for $q \leq 6$. This can be performed by inserting all the possibilities for q and solving the quadratic in p, say for $q = 6$ we get $p^2 - 23p + 142 \leq 0$ without any solution.

Henceforth, we assume that $q > 6$. (Then we also have $p > 6$.) We recall from the beginning of the proof that $d_2 \leq q + 2$.

Suppose first that $d_2 = q + 2$, and let $d_1 = p - 1$. Solving the system (5)–(6), we obtain

$$t_3 = \frac{1}{2}p(17-3p) + q(3q-8) - 7,\ \ t_4 = \frac{1}{2}\big(p(p-5) - 2q(q-2) + 4\big).$$

We claim that t_3 and t_4 cannot simultaneously be non-negative. Indeed, if we assume that both are non-negative, then from $t_4 \geq 0$, we conclude that q does not exceed the largest root of the quadratic (in q) in the previous expression for t_4. In other words, $q \leq \frac{1}{2}(\sqrt{2(p^2-5p+6)}+2)$. On the other hand, since t_3 increases with q, the largest t_3 (such that $t_4 \geq 0$) is obtained by equating q with the previous upper bound. Substituting for such a q, we obtain $t_3 \leq p - 3 - \sqrt{2(p-2)(p-3)} < 0$, but this contradicts the assumption that $t_3 \geq 0$.

Let now $d_1 = p - i$ for $i \geq 2$. Considering (5) and (6) we get that t_3 increases with each of d_1 and q. In addition, we have $q + 2 = d_2 \leq d_1 = p - i$, i.e. $q \leq p - i - 2$. Therefore, t_3 attains its maximum for $(d_1, q) = (p-2, p-4)$. Under this condition, (5)–(6) gives $t_3 = 52 - 12p < 0$ for $p > 6$, which is a contradiction.

Supposing that $d_2 = q + 1$, we get that t_3 attains its maximum for $(d_1, q) = (p-1, p-2)$, which as before leads to the impossible scenario $t_3 = 10 - 3p < 0$ for $p > 6$.

Supposing that $d_2 = q - i$ for $i \geq 0$ and, as before, setting $d_1 = p - 1$, from (5)–(6) we obtain $t_3 = -\frac{1}{2}\big((p-2)(3p-11) + i(i^2 - i(3q-7) + q(3q-14) + 14)\big) < 0$, where the inequality follows since $p > 6$ and the discriminant of the quadratic in i is negative for $q > 6$. This establishes a final contradiction.

Case 2: $d_1 = p$. From (5) and (6) we compute

$$t_3 = \frac{1}{2}(d_2 - q)\big(d_2^2 + (q-7)(d_2+q) + 14\big).$$

It holds $d_2 \leq q$, as otherwise the number of triangles in Line(T) would be less than the number of triangles in Line(G). For $d_2 < q$, we immediately obtain $t_3 < 0$. For $d_2 = q$ we have $t_3 = 0$ and (from (6)) $t_4 = 0$, and so $d_3 \leq 2$.

So far, we have $d_1 = p, d_2 = q$ and $d_3 \leq 2$. Taking into account that the number of pendant vertices in every tree is equal to $2 + \sum_{d_i \geq 3}(d_i - 2)$, we arrive at $\deg(T) = \deg(G)$. □

Lemma 3.2. *Let H be obtained from a tree G by deleting a pendant vertex and subdividing an edge which is not on a hanging path. Then $\mu_1(H) < \mu_1(G)$.*

Proof. This result is a direct consequence of Theorem 2.4. □

We are ready to prove our main result. The main ingredient is Lemma 3.1 which gives vertex degrees of G. The proof also relies on a multiple application of Lemma 3.2.

Proof of Theorem 1.1. By Lemma 3.1, G is a tree with the same vertex degree collection, and so $G \cong M(a', b', c', p, q)$. It remains to show that at least one of inequalities of (1) fails to hold. By way of contradiction, assume that all three hold. Denote the vertices of degree p and q in G by u and v, respectively (so, as in Fig. 1(b)).

We proceed with a successive application of Lemma 3.2 each time to a pendant vertex of some of $p-1$ hanging paths attached at u or some of $q-1$ hanging paths attached at v and an edge between u and v until the length of exactly one hanging path attached at u is a, the length of exactly one hanging path attached at v is c and the length of the remaining hanging paths is 1 (that is, they can be seen as pendant vertices); clearly, this can be performed under the assumption on inequalities of (1). In this way we have transformed G into $T \cong D(a, b, c, p, q)$, along with $\mu_1(G) > \mu_1(T)$ (this inequality follows from Lemma 3.2), which means that T and G are not Laplacian cospectral. Hence, we have established a contradiction, and the proof is complete. □

4. Consequences

In the forthcoming Theorems 4.1 and 4.5 we resolve some particular cases in which a double starlike tree is determined by its Laplacian spectrum. Each of these theorems can be seen as a consequence of Theorem 1.1.

Theorem 4.1. *The following double starlike trees are determined by the Laplacian spectrum:*

(i) ([6, 8]) $D(1, b, 1, p, q)$,

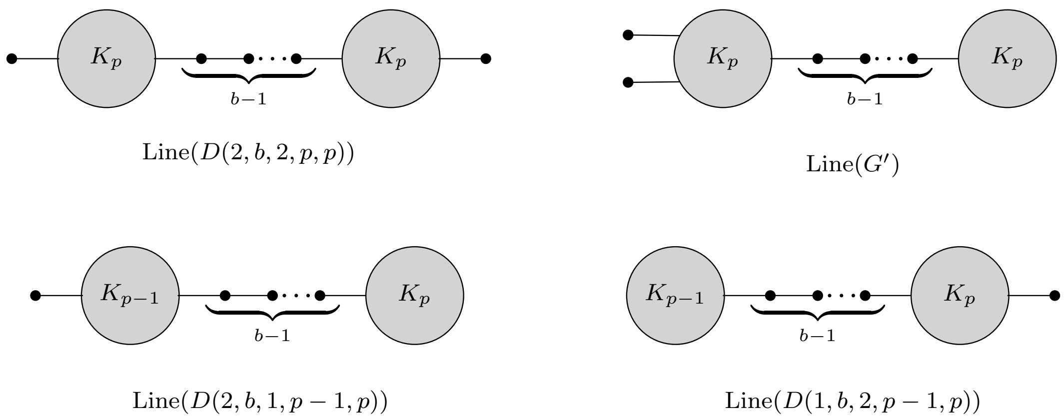

Figure 2: Line graphs considered in the proof of Lemma 4.3.

(ii) $D(1,b,2,p,p)$.

Proof. Assume that a tree T under consideration is Laplacian cospectral with some graph G. Then G is as in Theorem 1.1.

(i): By definition of $M(a',b',c',p,q)$, we have $a',c' \geq 1$, and so $a' \geq a, c' \geq c$. Since G and T have the same number of vertices, we also have $b' \leq b$. Hence, all the inequalities of Theorem 1.1 are satisfied, which declines the existence of G.

(ii): For $b' > b$, there must be $b' = b+1$, and then $a' = c' = 1$, but this leads to $G \cong D(1,b+1,1,p,p)$ which is impossible since, by (i), this tree is determined by the Laplacian spectrum. Hence, $b' \leq b$. If $c' < c$, then $c' = 1$ which leads to $a' \geq 2$. Since $p = q$, we can interchange the values of a' and c' to obtain $a' = a$ and $c' \geq c$, and the result follows from Theorem 1.1. ☐

To prove Theorem 4.5 we need some lemmas. We denote by $G(\ell,p,q)$ the graph obtained by inserting a path of length ℓ between a vertex of the complete graph K_p and a vertex of the complete graph K_q. In particular, $G(0,p,q)$ is the graph in which a vertex of K_p is identified with a vertex of K_q.

Lemma 4.2. *The characteristic polynomial of the adjacency matrix of $G(\ell,p,q)$ is*

(i) $\Phi_{G(0,p,q)}(x) = \Phi_{K_{p-1}}(x)\Phi_{K_{q-1}}(x)\big(x - (p-1)/(x-p+2) - (q-1)/(x-q+2)\big)$,

(ii) $\Phi_{G(1,p,q)}(x) = \Phi_{K_p}(x)\Phi_{K_q}(x) - \Phi_{K_{p-1}}(x)\Phi_{K_{q-1}}(x)$,

(iii) $\Phi_{G(2,p,q)}(x) = x\Phi_{K_p}(x)\Phi_{K_q}(x) - \Phi_{K_{p-1}}(x)\Phi_{K_q}(x) - \Phi_{K_p}(x)\Phi_{K_{q-1}}(x)$,

(iv) $\Phi_{G(\ell,p,q)}(x) = \Phi_{K_p}(x)\big(\Phi_{K_q}(x)\Phi_{P_{\ell-1}}(x) - \Phi_{K_{q-1}}(x)\Phi_{P_{\ell-2}}(x)\big) - \Phi_{K_{p-1}}(x)\big(\Phi_{K_q}(x)\Phi_{P_{\ell-2}}(x) - \Phi_{K_{q-1}}(x)\Phi_{P_{\ell-3}}(x)\big)$, *for* $\ell \geq 3$.

Proof. Item (i) follows from the formula for the characteristic polynomial of a cone [3, Theorem 2.1.7]. The remaining items are obtained by application of Theorem 2.7 to the first and the last edge of the internal path of $G(\ell,p,q)$. ☐

Henceforth, we denote by G' the graph $M(2,b,1,p,p)$ in which exactly 2 hanging paths attached at u are of length 2, while the remaining ones and those attached at v are of length 1.

Lemma 4.3. *It holds* $\lambda_1\big(\mathrm{Line}(D(2,b,2,p,p))\big) < \lambda_1\big(\mathrm{Line}(G')\big)$.

Proof. To make the reading easier, in Fig. 2 we have sketched four line graphs we are dealing with; a circle denotes the complete graph, for $b = 0$ a vertex of one complete graph is identified with a vertex of the other one, and for $b \geq 1$ the number of vertices between complete graphs is $b - 1$.

Applying Theorem 2.7 to a pendant edge of $\mathrm{Line}(D(2, b, 2, p, p))$ and to a pendant edge of $\mathrm{Line}(G')$, we obtain

$$\begin{aligned}\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) &= x\Phi_{\mathrm{Line}(D(1,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(D(1,b,2,p-1,p))}(x),\\ \Phi_{\mathrm{Line}(G')}(x) &= x\Phi_{\mathrm{Line}(D(1,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(D(2,b,1,p-1,p))}(x),\end{aligned}$$

which gives $\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(G')}(x) = \Phi_{\mathrm{Line}(D(2,b,1,p-1,p))}(x) - \Phi_{\mathrm{Line}(D(1,b,2,p-1,p))}(x)$. Repeating the procedure to the pendant edges of $\mathrm{Line}(D(2, b, 1, p-1, p))$ and $\mathrm{Line}(D(1, b, 2, p-1, p))$, we obtain $\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(G')}(x) = \Phi_{\mathrm{Line}(D(1,b,1,p-1,p-1))}(x) - \Phi_{\mathrm{Line}(D(1,b,1,p-2,p))}(x)$. In the notation introduced upon Lemma 4.2, this becomes

$$\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(G')}(x) = \Phi_{G(b,p-1,p-1)}(x) - \Phi_{G(b,p-2,p)}(x).$$

Using Lemma 4.2(i)–(iii) we compute

$$\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(G')}(x) = \begin{cases} (x+2)(x+1)^{2(p-3)}, & b = 0,\\ x(x+2)(x+1)^{2(p-3)}, & b = 1,\\ (x^2+x-2)(x+1)^{2p-5}, & b = 2,\end{cases}$$

which means that $\Phi_{\mathrm{Line}(D(2,b,2,p,p))}(x) - \Phi_{\mathrm{Line}(G')}(x) > 0$ for $x > 1$. Since $\lambda_1(\mathrm{Line}(G')) > 2$ (see [10, p. 63]), the previous inequality holds for $x = \lambda_1(\mathrm{Line}(G'))$, which proves (i) for $b \leq 2$. For $b \geq 3$, by setting $x = (t+1)/\sqrt{t}$, we obtain

$$\Phi_{P_b}\Big(\frac{t+1}{\sqrt{t}}\Big) = t^{-b/2}\frac{t^{b+1}-1}{t-1},$$

see [3, p. 47]. On the basis of Lemma 4.2(iv), we compute

$$\begin{aligned}\Phi_{\mathrm{Line}(D(2,b,2,p,p))}\Big(\frac{t+1}{\sqrt{t}}\Big) - \Phi_{\mathrm{Line}(G')}\Big(\frac{t+1}{\sqrt{t}}\Big) &= \Phi_{G(b,p-1,p-1)}\Big(\frac{t+1}{\sqrt{t}}\Big) - \Phi_{G(b,p-2,p)}\Big(\frac{t+1}{\sqrt{t}}\Big)\\ &= \frac{(\sqrt{t}+1)^2 t^{-b/2}(t^{b+1}-1)\big(1+(1+t)/\sqrt{t}\big)^{2(p-2)}}{(t-1)(1+\sqrt{t}+t)^2}.\end{aligned}$$

The last fraction is positive for every $x = (t+1)/\sqrt{t} > 2$, which can be seen by taking into account $t > 0$ and $(t^{b+1}-1)/(t-1) > 0$. In particular, the latter conclusion holds for $x = \lambda_1(\mathrm{Line}(G'))$, which leads to the desired result. □

Lemma 4.4. *For $b' \leq b$, the graphs $G \cong D(2, b, 2, p, p)$ and $H \cong D(b-b'+3, b', 1, p, p)$ are not Laplacian cospectral.*

Proof. We use Theorem 2.8 to compute the difference $c_{n-4}(G) - c_{n-4}(H)$; clearly, if G and H are Laplacian cospectral, then this difference must be zero. We first assume that $b' \geq 1$. Since G and H have the same number of vertices and the same vertex degrees, we have

$$c_{n-4}(G) - c_{n-4}(H) = -40\big(M_2(G) - M_2(H)\big) + \frac{39}{2}\big(EM_1(G) - EM_1(H)\big).$$

For G, there are two edges uv with $\{d_u, d_v\} = \{1, 2\}$, $b-1$ edges with $d_u = d_v = 2$, $2(p-2)$ edges with $\{d_u, d_v\} = \{1, p\}$ and four edges with $\{d_u, d_v\} = \{2, p\}$. This gives

$$\begin{aligned} M_2(G) &= 2((p+2)p + 2b), \\ EM_1(G) &= 2\big(((p-2)p+5)p + 2b - 3\big). \end{aligned} \tag{7}$$

Similarly, H has one edge uv with $\{d_u, d_v\} = \{1, 2\}$, b edges with $d_u = d_v = 2$, $2p-3$ edges with $\{d_u, d_v\} = \{1, p\}$ and three edges with $\{d_u, d_v\} = \{2, p\}$, which gives

$$\begin{aligned} M_2(H) &= (2p+3)p + 4b + 2, \\ EM_1(H) &= (2p-3)(p-1)^2 + 3p^2 + 4b + 1. \end{aligned}$$

Now, we compute $c_{n-4}(G) - c_{n-4}(H) = 2 - p$. Indeed, all linear terms of b and non-linear terms of p are eliminated. Accordingly, G and H do not share the same Laplacian spectrum, unless $p = 2$, but the latter possibility is eliminated at the beginning of the paper, as for $p = 2$ both trees reduce to isomorphic paths.

It remains to consider the case $b' = 0$. Here we have a bit different structure of H, since the vertices of degree p are adjacent. Accordingly, H has one edge uv with $\{d_u, d_v\} = \{1, 2\}$, $b+1$ edges with $d_u = d_v = 2$, $2p-3$ edges with $\{d_u, d_v\} = \{1, p\}$, one edge with $\{d_u, d_v\} = \{2, p\}$ and one edge with $d_u = d_v = p$. Hence,

$$\begin{aligned} M_2(H) &= (3p-1)p + 4b + 6, \\ EM_1(H) &= (2p-3)(p-1)^2 + (5p-8)p + 4b + 9. \end{aligned} \tag{8}$$

Now, if $b \geq 1$, on the basis of (7) and (8), we obtain $c_{n-4}(G) - c_{n-4}(H) = (p-2)(p-3)$. The case $p = 2$ is not interesting to us. For $p = 3$, if $b = 1$ then G has exactly 9 vertices, and our statement is confirmed by a direct computation. Assume that for some $b \geq 2$, G and H are Laplacian cospectral. By Theorem 2.6, $\mathrm{Line}(G)$ and $\mathrm{Line}(H)$ are cospectral (with respect to the adjacency matrix). Since $b \geq 2$, $\mathrm{Line}(G)$ consists of 2 copies of a triangle with a pendant vertex along with a path of length at least 2 inserted between a vertex of degree 2 in one copy and the same vertex in the other. By removing any vertex of the internal path we get a disconnected subgraph, say G', with $\lambda_2(G') > 2$; indeed, both components of G' contain a triangle with at least one hanging path (whose largest eigenvalue is greater than 2, see [10, p. 63]) which leads to the desired conclusion. By the eigenvalue interlacing, we have $\lambda_2(\mathrm{Line}(G)) \geq \lambda_2(G') > 2$. On the other hand, since $b' = 0$, $\mathrm{Line}(H)$ consists of two triangles that have a common vertex, say u, and a hanging path attached at a vertex of degree 2 of one of these triangles. By removing u, the graph falls apart into a subgraph, say H', consisting of two paths, which gives $\lambda_1(H') < 2$. By the interlacing we have $\lambda_2(\mathrm{Line}(H)) \leq \lambda_1(H') < 2$, and therefore $\lambda_2(\mathrm{Line}(G)) > \lambda_2(\mathrm{Line}(H))$, which contradicts the assumption on cospectrality.

Finally, for $b = 0$, we have $M_2(G) = 3p^2 + 4, EM_1(G) = 2(p^3 - p^2 + p + 1)$. Together with equalities (8) in which we set $b = 0$, this leads to $c_{n-4}(G) - c_{n-4}(H) = 2 - p$, and we are done. □

Finally, we prove the following result. One part of the proof is based on the application of Lemma 3.2 in conjunction with Lemma 4.3, and the other part is based on Lemma 4.4.

Theorem 4.5. *The double starlike tree $D(2, b, 2, p, p)$ is determined by the Laplacian spectrum.*

Proof. Assume that a graph G is Laplacian cospectral with $D(2, b, 2, p, p)$. By Theorem 1.1, $G \cong M(a', b', c', p, p)$, for some choice of a', b', c' such that at least one of the following holds: $a' = 1, c' = 1, b' > b$. For $b' > b$ we have $G \cong D(1, b', 1, p, p)$ or $G \cong D(1, b', 2, p, p)$, but this is impossible since both graphs are determined by the Laplacian spectrum by Theorem 4.1. Suppose that $b' \leq b$ and, without loss of generality, $c' = 1$. In this case, all paths attached at v of G are pendant vertices (see Fig. 1(b)). We distinguish two cases.

Case 1: At least two paths attached at u are not pendant vertices. If the length of exactly two paths attached at u is 2 and the remaining ones are pendant vertices, then $G \cong G'$, where G' is defined upon Lemma 4.3. Otherwise, by a successive application of Lemma 3.2, we can transform G into G', along with $\lambda_1(\mathrm{Line}(G)) > \lambda_1(\mathrm{Line}(G'))$. In both scenarios, by virtue of Lemma 4.3, we have $\lambda_1(\mathrm{Line}(G)) > \lambda_1(\mathrm{Line}(D(2, b, 2, p, p)))$, which by Theorem 2.6 leads to $\mu_1(G) > \mu_1(D(2, b, 2, p, p))$. Hence, a contradiction.

Case 2: Exactly one path attached at u is not a pendant vertex. In this situation, the desired result follows from Lemma 4.4. □

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