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The Shape Operator of Real Hypersurfaces in $S^6(1)$

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Abstract: The aim of the paper is to present two results concerning real hypersurfaces in the six-dimensional sphere $S^6(1)$. More precisely, we prove that real hypersurfaces with the Lie-parallel shape operator A must be totally geodesic hyperspheres. Additionally, we classify real hypersurfaces in a nearly Kähler sphere $S^6(1)$ whose Lie derivative of the shape operator coincides with its covariant derivative.

Keywords: Hopf hypersurfaces; nearly Kähler manifolds; real hypersurfaces; shape operator; Lie derivative

MSC: 53B25; 53B35; 53B21

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1. Introduction

An almost Hermitian manifold with an almost complex structure J and the Levi-Civita connection $\tilde{\nabla}$ is, respectively, Kähler or nearly Kähler if the tensor field $G(X,Y) = (\tilde{\nabla}_X J)Y$ is vanishing or skew-symmetric. In [1], it was shown that an arbitrary nearly Kähler manifold can be locally decomposed into manifolds of three particular types, with six-dimensional nearly Kähler manifolds being one of those types. We recall that it has been shown by Butruille, see [2], that only four homogeneous, six-dimensional, strictly nearly Kähler manifolds exist: the six-dimensional sphere $S^6(1)$, the manifold $S^3 \times S^3$, the projective space CP^3, and the flag manifold $SU(3)/U(1) \times U(1)$. We note that, out of these four, only the sphere $S^6(1)$ is endowed with standard metrics.

If we denote by N the unit normal vector field on the hypersurface M of an almost Hermitian manifold, the tangent vector field $\xi = -JN$ is called the Reeb vector field or characteristic vector field. We denote by g the metric on $S^6(1)$ induced by the standard Euclidean metric $\langle , \rangle$ in the space $\mathbb{R}^7$. The shape operator of real hypersurfaces M of $S^6(1)$ is denoted by A and satisfies $g(AX,Y) = g(h(X,Y),N)$ for all X, Y tangent to M, where h is the second fundamental form of M.

We say that a hypersurface M is Hopf if the vector field ξ satisfies $A\xi = \alpha\xi$ for a certain differentiable function α, and then ξ is a principal vector field. We also note that the function α is locally constant, see [3]. It was shown in [3] that a connected Hopf hypersurface of a nearly Kähler $S^6(1)$ is an open part of either a geodesic hypersphere or a tube around an almost complex curve in $S^6(1)$. Therefore, at each point of a Hopf hypersurface in $S^6(1)$, there exist either one, two, or three different principal curvatures.

The parts of totally geodesic hyperspheres are, of course, umbilical and have only one principal curvature α. All other Hopf hypersurfaces in $S^6(1)$ are parts of tubes around almost complex curves in $S^6(1)$, and the principal curvature α has a multiplicity of 3. If the almost complex curve is totally geodesic, there is only one other principal curvature μ that has a multiplicity of 2. If the almost complex curve is not totally geodesic, then it is one of types (I), (II), or (III), and α has two other principal curvatures, μ and λ, that have a multiplicity of 1. For details, we refer the reader to [4].

In recent years, the study of Riemannian hypersurfaces in different ambient spaces endowed with an almost complex structure has been an active field of research. In particular, many studies deal with the question of the existence and classification of hypersurfaces that satisfy various conditions related to their parallelism, particularly related to its shape operator, such as η-parallelness or pseudo-parallelness; we refer the reader to [5,6]. The non-existence of real hypersurfaces in complex space forms with a parallel shape operator was observed in [7]. Additionally, Kimura and Maeda [8] classified real hypersurfaces in complex projective space whose shape operator is ξ-parallel, i.e., $\nabla_\xi A = 0$.

Recall that Lie derivative $\mathcal{L}$ of a vector field on M is given by $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$, while the Lie derivative of the shape operator is given by $(\mathcal{L}_X A)Y = \nabla_X(AY) + A\nabla_Y X - \nabla_{AY} X - A\nabla_X Y$ for the X, Y tangent of M. The Lie derivative, with respect to the vector field, has many applications in physics, in particular in mechanics, hydrodynamics, theory of relativity, and cosmology. Hence, hypersurfaces whose shape operator is invariant or has at least a certain regularity in its behavior with respect to the Lie derivative are of particular interest. Ki, Kim, and Lee, see [9], classified real hypersurfaces in complex space forms whose shape operator is Lie ξ-parallel, i.e., $\mathcal{L}_\xi A = 0$. Suh, in [10], provides a characterization of real hypersurfaces of type A in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, which are tubes over totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of the vanishing Lie derivative of the shape operator A along the direction of the Reeb vector field ξ. In [11], the authors prove the nonexistence of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, whose shape operator satisfies the relation $\mathcal{L}_X A = \nabla_X A$, from the X orthogonal to ξ.

Motivated by these results, we consider a similar line of research regarding the nearly Kähler sphere S^6 and prove the following theorems.

Theorem 1. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M is Lie-parallel, i.e., $\mathcal{L}_X A = 0$, $X \in TM$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

Theorem 2. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

Additionally, given that totally geodesic hyperspheres also satisfy the stronger condition $\mathcal{L}_X A = \nabla_X A$, for all $X \in TM$, we obtain the following:

Corollary 1. *Let M be a real hypersurface in $S^6(1)$. The shape operator A on M satisfies $\mathcal{L}_X A = \nabla_X A$, for all $X \in TM$, if and only if M is a totally geodesic hypersphere in $S^6(1)$.*

In particular, we note that most of the known results dealing with this type of problem regard hypersurfaces of Kähler manifolds. In this case, the almost complex structure is parallel, making it easier to conduct calculations. For hypersurfaces of the nearly Kähler sphere S^6, the covariant derivative of the almost complex structure, tensor G, is skew-symmetric, which, in a technical sense, imposes the need to approach the problem in a different manner, see [12,13]. As a consequence, we need to use a suitable moving frame along the hypersurface, which is nicely suited to the given structure, in order to obtain its properties and analyze them.

2. Preliminaries

First, we provide a brief exposition of how the standard nearly Kähler structure J on $S^6(1)$ arises in a natural manner from the multiplication of Cayley numbers $\mathcal{O}$. A vector cross product $\times$ given as follows:

$$u \times v = \frac{1}{2}(uv - vu),$$

and is well defined on the space of purely imaginary Cayley numbers $\mathcal{O}$, which we may identify with $\mathbb{R}^7$.

Then, vectors $e_1, ...e_7$ of the standard orthonormal basis of the space $\mathbb{R}^7$ satisfy the relations provided in the following table of multiplication.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

Any orthonormal basis or frame for which the relations of this table hold is called a G_2 basis or frame. Then, for an arbitrary point $p \in S^6(1)$ and $X \in T_pS^6(1)$, the (1,1)-tensor field J is defined by

$$J_pX = p \times X$$

and is an almost complex structure.

We will denote by $\langle , \rangle$ the standard metric in the space $\mathbb{R}^7$ and, by g, the induced metric on $S^6(1)$. Further, we denote D and $\bar{\nabla}$ by the corresponding Levi–Civita connections.

Let M be a Riemannian submanifold of the sphere $S^6(1)$ with a Hermitian structure (J, g). The tensor field G of type $(2, 1)$, defined by $G(X,Y) = (\bar{\nabla}_X J)Y$, is skew symmetric, which makes the almost complex structure a nearly Kähler one. The tensor field G has the following properties:

$$G(X, JY) + JG(X,Y) = 0, \qquad g(G(X,Y), Z) + g(G(X, Z), Y) = 0.$$

Additionally, for tangent vector fields X, Y, and Z, see [14], it holds that

$$(\bar{\nabla}G)(X,Y,Z) = g(X, Z)JY - g(X,Y)JZ - g(JY, Z)X. \tag{1}$$

Let ∇ and $\nabla^\perp$ be, respectively, the Levi–Civita connection of the submanifold M and the normal connection in the normal bundle $T^\perp M$ of M in $S^6(1)$. The formulae of Gauss and Weingarten for the hypersurface are as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X N = -AX + \nabla^\perp_X N,$$

where X and Y are tangent vector fields and N is a normal vector field on the hypersurface. Here, A and h denote the shape operator and the second fundamental form. The relationship between them is given by $g(h(X,Y), N) = g(AX,Y)$. The Gauss equation for the hypersurface is as follows:

$$\begin{aligned} R(X,Y,Z,W) =& g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &+ g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)),\ X,Y,Z,W \in TM, \end{aligned} \tag{2}$$

where we denote by R the Riemannian curvature tensor of M.

Now, let M be a hypersurface in $S^6(1)$. Using the almost complex structure J on $S^6(1)$, the normal vector field N on M, we define the corresponding Reeb vector field $\xi = -JN$ with dual 1-form $\eta(X) = g(X, \xi)$. Then, $\mathcal{D} = \operatorname{Ker} \eta = \{X \in TM \mid \eta(X) = 0\}$ is a four-dimensional almost complex distribution on M.

3. The Moving Frame for Hypersurfaces in $S^6(1)$

Now, we will present the construction of one of the local moving frames that is, roughly speaking, compatible with the structure on the hypersurface, and hence, easier to work with. Also, we will present the relationship between the connection coefficients in this particular frame, for more details see [12,15].

For any unit vector field $E_1 \in \mathcal{D}$, let $E_2 = JE_1$, $E_3 = G(E_1, \xi)$, and $E_4 = JE_3$. Then, as shown in [15], the set $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ is a local orthonormal moving frame. One such frame is uniquely determined by the choice of the vector field $E_1 \in \mathcal{D}$.

We note that, additionally, the following relations are also valid for each frame.

Lemma 1 ([15])**.** *The orthonormal frame $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ satisfies the following relations:*

$$\begin{array}{llll} G(E_1, E_2) = 0, & G(E_1, E_3) = -\xi, & G(E_1, E_4) = N, & G(E_1, \xi) = E_3, \\ G(E_1, N) = -E_4, & G(E_2, E_3) = -N, & G(E_2, E_4) = \xi, & G(E_2, \xi) = -E_4, \\ G(E_2, N) = -E_3, & G(E_3, E_4) = 0, & G(E_3, \xi) = -E_1, & G(E_3, N) = E_2, \\ G(E_4, \xi) = E_2, & G(E_4, N) = E_1. & & \end{array} \tag{3}$$

We further denote the coefficients of the covariant derivatives in the given frame as follows:

$$g_{ij}^k = g(D_{E_i} E_j, E_k), \qquad h_{ij} = g(D_{E_i} E_j, N), \qquad 1 \leq i, j, k \leq 5. \tag{4}$$

Since D is a metric connection and the second fundamental form is symmetric, straightforwardly, we obtain $g_{ij}^k = -g_{ik}^j$, and $h_{ij} = h_{ji}$.

Using the definition of $G(X, Y) = (\bar{\nabla}_X J)Y$ and the expression for its covariant derivative, we obtain, the following two lemmas, see [12].

Lemma 2 ([12])**.** *For the previously defined coefficients, we obtain the following:*

$$\begin{array}{lllll} g_{12}^3 = -g_{11}^4, & g_{12}^4 = g_{11}^3, & h_{11} = -g_{12}^5, & h_{12} = g_{11}^5, & g_{22}^3 = -g_{21}^4, \\ g_{22}^4 = g_{21}^3, & g_{22}^5 = -g_{11}^5, & h_{22} = g_{21}^5, & g_{32}^3 = -g_{31}^4, & g_{32}^4 = g_{31}^3, \\ h_{13} = 1 - g_{32}^5, & h_{23} = g_{31}^5, & g_{42}^3 = -g_{41}^4, & g_{42}^4 = g_{41}^3, & h_{14} = -g_{42}^5, \\ h_{24} = -1 + g_{41}^5, & g_{52}^3 = -1 - g_{51}^4, & g_{52}^4 = g_{51}^3, & h_{15} = -g_{52}^5, & h_{25} = g_{51}^5, \\ g_{32}^5 = 2 + g_{14}^5, & g_{42}^5 = -g_{13}^5, & g_{31}^5 = -g_{24}^5, & g_{41}^5 = 2 + g_{23}^5, & h_{33} = -g_{43}^5, \\ h_{34} = g_{33}^5, & g_{44}^5 = -g_{33}^5, & h_{44} = g_{43}^5, & h_{35} = -g_{54}^5, & h_{45} = g_{53}^5. \end{array}$$

Proof. By taking $X \in \{E_1, ..., \xi\}$ and $Y \in \{E_1, ..., \xi, N\}$ in the relation

$$\bar{\nabla}_X(JY) = G(X, Y) + J(\bar{\nabla}_X Y),$$

and using (3), we obtain the proof of the lemma.

For $X = E_1$ and $Y = E_1$, we obtain the following:

$$g_{12}^3 = -g_{11}^4, \qquad g_{12}^4 = g_{11}^3, \qquad h_{11} = -g_{12}^5, \qquad h_{12} = g_{11}^5.$$

For $X = E_2$ and $Y = E_1$, we obtain the following:

$$g_{22}^3 = -g_{21}^4, \qquad g_{22}^4 = g_{21}^3, \qquad g_{22}^5 = -h_{12} = -g_{11}^5, \qquad h_{22} = g_{21}^5.$$

In a similar manner, we obtain other relations. □

Lemma 3 ([12])**.** *For the coefficient (4), the following holds:*

$$\begin{array}{lll} g_{52}^5 = g_{11}^2 + g_{13}^4, & g_{51}^5 = -g_{21}^2 - g_{23}^4, & g_{54}^5 = g_{31}^2 + g_{33}^4, \\ g_{53}^5 = -g_{41}^2 - g_{43}^4, & h_{55} = -g_{51}^2 - g_{53}^4. & \end{array} \tag{5}$$

Proof. By taking $X = E_1$, $Y = E_1$, and $Z = E_3$ in (1), we obtain the following: $g_{52}^5 = g_{11}^2 + g_{13}^4$. Similarly, by taking $X = E_2$, $Y = E_1$, and $Z = E_3$, we obtain $g_{51}^5 = -g_{21}^2 - g_{23}^4$; for $X = E_3$, $Y = E_1$, and $Z = E_3$, we obtain $g_{54}^5 = g_{31}^2 + g_{33}^4$. Finally, for $X = E_4$, $Y = E_1$, and $Z = E_3$ and $X = \xi$, $Y = E_1$, and $Z = E_3$, respectively, we get $g_{53}^5 = -g_{41}^2 - g_{43}^4$ and $h_{55} = -g_{51}^2 - g_{53}^4$. Straightforward computation shows that (1) is then satisfied for an arbitrary choice of vector fields which completes the proof. □

Recall that we still have the possibility of choosing $E_1 \in \mathcal{D}$. We can decompose the vector field $A\xi$ into an orthogonal sum of two vector fields, parallel to $\mathcal{D}$ and ξ, respectively. Let E_1 be the unit vector field collinear with the first component. Then, we can write

$$A\xi = \beta E_1 + \alpha\xi, \tag{6}$$

for differentiable functions α and β. Moreover, by choosing the direction of E_1, we may assume that $\beta \geq 0$. Of course, in the case of the Hopf hypersurface, when $\beta = 0$, the vector field E_1 is still not uniquely determined by this.

Since $A\xi$ has no components in the direction of vector fields E_2, E_3, E_4, causing it to vanish, we obtain:

$$g_{13}^4 = -g_{11}^2 - \beta, \quad g_{23}^4 = -g_{21}^2, \quad g_{33}^4 = -g_{31}^2, \quad g_{43}^4 = -g_{41}^2, \quad g_{53}^4 = -g_{51}^2 - \alpha.$$

From the Gauss equation we obtain the next two relations between the coefficients. By implementing $X = E_2$, $Y = E_3$, $Z = E_1$, and $W = E_2$ into the Gauss Equation (2) we obtain the following:

$$\begin{aligned} E_2(g_{31}^2) =& g_{21}^2 g_{21}^4 - 3g_{21}^5 - 2g_{14}^5 g_{21}^5 + 2g_{11}^5 g_{24}^5 + g_{11}^2(-g_{21}^3 + g_{31}^2) - 2g_{21}^4 g_{31}^3 \\ &+ 2g_{21}^3 g_{31}^4 + g_{31}^2 g_{31}^4 - g_{21}^2 g_{41}^2 - g_{31}^3 g_{41}^2 - 2g_{51}^2 - g_{14}^5 g_{51}^2 + g_{23}^5 g_{51}^2 + E_3(g_{21}^2), \end{aligned}$$

and then by implementing $(X, Y, Z, W) = (E_2, E_3, E_3, E_4)$ in (2), we also obtain the following:

$$\begin{aligned} &(3 + 2g_{14}^5)g_{21}^5 - 2g_{24}^5(g_{11}^5 + g_{33}^5) + (1 + 2g_{23}^5)g_{34}^5 + (-2 - g_{14}^5 + g_{23}^5)\alpha \\ &+ (-g_{21}^3 + g_{31}^2)\beta = 0. \end{aligned} \tag{7}$$

4. Proof of the Theorem 1

Let us denote $L_{ij}^k = g((\mathcal{L}_{E_i} A)E_j, E_k)$. The condition that the shape operator A is Lie-parallel is equivalent to $L_{ij}^k = 0, 1 \leq i, j, k \leq 5$.

Lemma 4. *Let M be a hypersurface in $S^6(1)$ with Lie ξ-parallel shape operator A. Then, M is a Hopf hypersurface.*

Proof. Suppose that M is a non-Hopf hypersurface, i.e., $\beta \neq 0$. From $L_{55}^i = 0, 1 \leq i \leq 4$, we have, respectively,

$$\xi(\beta) = -g_{11}^5\beta, \qquad g_{51}^2 = -g_{12}^5, \qquad g_{51}^3 = -g_{13}^5, \qquad g_{51}^4 = -g_{14}^5.$$

Using this, from

$$\begin{aligned} 0 &= L_{51}^5 = -3g_{11}^5\beta, \\ 0 &= L_{53}^5 = (-g_{13}^5 + 2g_{24}^5)\beta, \\ 0 &= L_{54}^5 = (-3 - g_{14}^5 - 2g_{23}^5)\beta, \end{aligned}$$

we get, respectively $g^5_{11} = 0$, $g^5_{13} = 2g^5_{24}$ and $g^5_{14} = -3 - 2g^5_{23}$. From $0 = L^2_{51} = 6(1 + g^5_{23})$, we obtain $g^5_{12} = -1$, and

$$0 = L^1_{52} = (g^5_{12} + g^5_{21})^2 + (g^5_{24})^2 + \beta^2$$

gives us a contradiction. □

Therefore, now we assume M to be a Hopf hypersurface, implying that $\beta = 0$ and α are constant.

Recall that, for Hopf hypersurfaces, we still have the option of choosing $E_1 \in \mathcal{D}$, and we can assume that E_1 is an eigenvector field for the shape operator A. As

$$AE_1 = -g^5_{12}E_1 + g^5_{11}E_2 - (1 + g^5_{14})E_3 + g^5_{13}E_4,$$

we obtain $g^5_{11} = 0$, $g^5_{14} = -1$, $g^5_{13} = 0$.

Now, from $0 = L^5_{22} = 2g^5_{24}$, we obtain $g^5_{24} = 0$, and then from $0 = L^5_{24} + L^5_{13} = -4g^5_{33}$, we obtain $g^5_{33} = 0$. Now let us see what the shape operator looks like:

$$\begin{aligned} &AE_1 = -g^5_{12}E_1, && AE_2 = g^5_{21}E_2 + (1 + g^5_{23})E_4, && AE_3 = -g^5_{34}E_3, \\ &AE_4 = (1 + g^5_{23})E_2 + g^5_{43}E_4, && AE_5 = \alpha E_5. && \end{aligned} \tag{8}$$

Lemma 5. *Let M be a Hopf hypersurface in $S^6(1)$ with Lie-parallel shape operator A. Then, the moving frame can be chosen so that both E_1 and E_3 are the eigenvectors of the shape operator A for the eigenvalues α.*

Proof. From (8), we want to prove that $-g^5_{12} = \alpha = -g^5_{34}$.

Let us first assume that $g^5_{34} \neq -\alpha$. From

$$\begin{aligned} 0 &= L^5_{23} = (1 - g^5_{23})(g^5_{34} + \alpha), \\ 0 &= L^3_{45} = -(g^5_{34} + \alpha)(g^5_{43} + g^2_{51} + \alpha), \\ 0 &= L^5_{43} = (g^5_{34} - g^5_{43})(g^5_{34} + \alpha), \end{aligned}$$

we obtain, respectively, $g^5_{23} = 1$, $g^2_{51} = -g^5_{43} - \alpha$, and $g^5_{43} = g^5_{34}$. Next, from $0 = L^3_{54} = -2g^4_{51}$ we have $g^4_{51} = 0$, and then, from $0 = L^4_{53} = -4(1 + (g^5_{34})^2)$, we get a contradiction. Hence, $g^5_{34} = -\alpha$.

Now, suppose that $g^5_{12} \neq -\alpha$. From

$$0 = L^5_{41} = -(3 + g^5_{23})(g^5_{12} + \alpha), \qquad 0 = L^1_{25} = -(g^5_{21} - g^2_{51})(g^5_{12} + \alpha),$$

we have $g^5_{23} = -3$ and $g^2_{51} = g^5_{21}$. Then, from $0 = L^1_{45} = -(-1 - g^4_{51})(g^5_{12} + \alpha)$, we obtain $g^4_{51} = -1$ and now, from $0 = L^2_{51} = -4 - (g^5_{12} + g^5_{21})^2$, we obtain a contradiction, which completes the proof of this lemma. □

Now, (7) becomes $g^5_{21} - (2 + g^5_{23})\alpha = 0$, so $g^5_{21} = (2 + g^5_{23})\alpha$, and we obtain the following:

$$0 = L^5_{12} = -(1 + g^5_{23})(3 + g^5_{23})(1 + \alpha^2).$$

If we assume that $g^5_{23} = -3$, then from $0 = L^5_{32} = 2g^5_{43} - 6\alpha$, we obtain $g^5_{43} = 3\alpha$, and from $0 = L^5_{34} = -8(1 + \alpha^2)$, we obtain a contradiction. Therefore, it must be $g^5_{23} = -1$.

From (8), we now see that the shape operator A of the hypersurface M has a quadruple eigenvalue α and a single g^5_{43}. Since M is a Hopf surface, it is possible that M is a totally geodesic hypersphere, i.e., $g^5_{34} = \alpha = 0$.

Straightforward computation shows that all L^k_{ij}, $1 \leq i, j, k \leq 5$ are equal to zero, which completes the proof.

5. Proof of the Theorem 2

Let us denote $a_{ij} = g(A\nabla_{E_i}\xi - \nabla_{AE_i}\xi, E_j)$. The condition $\mathcal{L}_\xi A = \nabla_\xi A$ is equivalent to $a_{ij} = 0, 1 \le i, j \le 5$.

Lemma 6. *Let M be a hypersurface in $S^6(1)$. If the shape operator A on M satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, then M is a Hopf hypersurface.*

Proof. Suppose that M is a non-Hopf hypersurface, i.e., $\beta \neq 0$. From $a_{i5} = 0, 1 \le i \le 4$, we obtain, respectively,

$$-g_{11}^5\beta = 0, \qquad -g_{21}^5\beta = 0, \qquad g_{24}^5\beta = 0, \qquad -(2 + g_{23}^5)\beta = 0,$$

so $g_{11}^5 = g_{21}^5 = g_{24}^5 = 0$ and $g_{23}^5 = -2$. Now, from $0 = a_{54} = (-1 + g_{14}^5)\beta$, we obtain $g_{14}^5 = 1$, and then, $a_{21} = -4 \neq 0$, which is a contradiction. □

Suppose now that M is a Hopf hypersurface, i.e., $\beta = 0$.

To determine the moving frame, we can choose E_1 to be an eigenvector field for the shape operator A. As

$$AE_1 = -g_{12}^5 E_1 + g_{11}^5 E_2 - (1 + g_{14}^5)E_3 + g_{13}^5 E_4,$$

we obtain $g_{11}^5 = 0$, $g_{14}^5 = -1$, $g_{13}^5 = 0$.

Now, from $0 = a_{22} = -2g_{24}^5$, we obtain $g_{24}^5 = 0$; then, we obtain $0 = a_{12} = g_{33}^5$. From $a_{14} = 0$ and $a_{32} = 0$, we obtain, respectively,

$$-g_{12}^5 g_{23}^5 + g_{43}^5 = 0, \qquad -g_{21}^5 - (2 + g_{23}^5)g_{34}^5 = 0,$$

so $g_{43}^5 = g_{12}^5 g_{23}^5$ and $g_{21}^5 = -(2 + g_{23}^5)g_{34}^5$. The only non-zero a_{ij} are now the following:

$$\begin{aligned}
a_{12} &= 1 - (g_{12}^5)^2 + g_{23}^5 + g_{12}^5(2 + g_{23}^5)g_{34}^5, \\
a_{21} &= [2 + g_{23}^5][1 + g_{23}^5 - g_{12}^5 g_{34}^5 + (2 + g_{23}^5)(g_{34}^5)^2], \\
a_{23} &= g_{23}^5(1 + g_{23}^5)(g_{12}^5 - g_{34}^5), \\
a_{34} &= -1 - g_{23}^5 - g_{34}^5(g_{12}^5 g_{23}^5 + g_{34}^5), \\
a_{41} &= (1 + g_{23}^5)(2 + g_{23}^5)(g_{12}^5 - g_{34}^5), \\
a_{43} &= g_{23}^5(1 + g_{23}^5 + (g_{12}5)^2 g_{23}^5 + g_{12}^5 g_{34}^5).
\end{aligned}$$

We begin further discussion from $a_{23} = g_{23}^5(1 + g_{23}^5)(g_{12}^5 - g_{34}^5) = 0$.

If we assume that $g_{23}^5 = 0$, then $a_{34} = 0$ becomes $-1 - (g_{34}^5)^2 = 0$, which is a contradiction. If we assume that $g_{12}^5 = g_{34}^5$, from $0 = a_{12} = (1 + (g_{12}^5)^2)(1 + g_{23}^5)$, we obtain $1 + g_{23}^5 = 0$, so it is enough to consider only the case $g_{23}^5 = -1$.

Now, $0 = a_{12} = g_{12}^5(g_{34}^5 - g_{12}^5)$ gives us $g_{12}^5 = 0$ or $g_{34}^5 = g_{12}^5$.

Assuming $g_{12}^5 = 0$, $a_{21} = 0$ becomes $(g_{34}^5)^2 = 0$, so $g_{34}^5 = 0$ and now all a_{ij} are zero. In this case, the shape operator A has a quadruple eigenvalue 0 and one is α. As M is a Hopf hypersurface, this is possible if and only if $\alpha = 0$, i.e., M is a totally geodesic hypersphere.

Assuming $g_{34}^5 = g_{12}^5$, we assume that all a_{ij} are zero. In this case, the shape operator A has a quadruple eigenvalue $-g_{12}^5$ and one is α. As M is a Hopf hypersurface, this is possible if and only if $-g_{12}^5 = \alpha = 0$, i.e., M is a totally geodesic hypersphere.

Hence, if the shape operator A on M satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, then M is a totally geodesic hypersphere.

If M is a totally geodesic hypersphere then the shape operator A on M obviously satisfies $\mathcal{L}_\xi A = \nabla_\xi A$, and that completes the proof.

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