Filomat 37:25 (2023), 8481–8488
https://doi.org/10.2298/FIL2325481S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Riemannian and sub-Riemannian structures on a cotangent bundle of Heisenberg group

Tijana Šukilović[a], Srdjan Vukmirović[a]

[a]*University of Belgrade, Faculty of Mathematics, Belgrade, Serbia*

Abstract. In this paper we give a classification of left invariant sub-Riemannian structures on cotangent bundle of $2n+1$ dimensional Heisenberg group T^*H_{2n+1}. We show that the sub-Riemannian metric is tamed by the corresponding Riemannian metric on T^*H_{2n+1}. We also describe Riemannian and sub-Riemannian geodesics on T^*H_{2n+1}.

Introduction

A sub-Riemannian geometry constitutes the natural generalization of Riemannian geometry. It has been a research focus since the 1980's, with motivations and ramifications in several parts of pure and applied mathematics such as control theory, classical and quantum mechanics, symplectic and contact geometry, analysis of hypoelliptic operators, univalent function theory, etc.

A sub-Riemannian manifold is a manifold endowed with a distribution and a fiber-inner product on it. Here by a distribution is considered a linear sub-bundle of the tangent bundle of the manifold, usually referred to as a horizontal space. The Chow's theorem asserts that for the completely non-integrable distribution on a connected manifold any two points can be connected by a horizontal path. The sub-Riemannian distance between two points is defined in the same manner as in the Riemannian case, except that it is only allowed to travel along the horizontal curves joining two points.

More on a sub-Riemannian geometry can be found in a wonderful book by Montgomery [10]. For readers preferring the approach from the optimal control theory, one should look at the book by Agrachev, Barilari and Boscain [1] and for the results focusing on Lie groups, a survey paper by Sachkov [11].

The simplest non-trivial example of sub-Riemannian geometry is the geometry of Heisenberg group. The Riemannian case of nilpotent Lie groups is very thoroughly investigated (for example, see [6–9, 14]). Sub-Riemannian geometry of the generalized Heisenberg group H_{2n+1} has been studied by various authors: from Vershik and Gershkovich [13] in the 1980s, to the more recent works of Biggs and Nagy [3, 4].

In this paper we will make further advances, we will investigate the Riemannian and sub-Riemannian geometry of the cotangent bundle of the generalized Heisenberg group that has natural structure of a Lie group.

2020 *Mathematics Subject Classification*. Primary 22E25; Secondary 53C17, 53C55.
Keywords. Cotangent bundle; Heisenberg group; Left invariant metrics; Sub-Riemannian structures; Geodesics.
Received: 13 November 2022; Accepted: 23 December 2022
Communicated by Mića Stanković and Zoran Rakić
This research was supported by the Science Fund of the Republic of Serbia, Grant No. 7744592, Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics - MEGIC.
Email addresses: tijana@matf.bg.ac.rs (Tijana Šukilović), vsrdjan@matf.bg.ac.rs (Srdjan Vukmirović)

Let G be a Lie group and $\mathcal{D} \subset \mathfrak{g}$ a linear subset of its corresponding Lie algebra. We identify the Lie algebra with the space of left-invariant vector fields on the Lie group, in which case $\mathcal{D}$ corresponds to a left-invariant distribution. The bracket-generating condition is that $\mathcal{D}$ generates $\mathfrak{g}$, meaning that the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathcal{D}$ is $\mathfrak{g}$ itself. The distribution satisfying the bracket-generating condition is completely non-integrable. Hence, the restriction of an inner product on $\mathfrak{g}$ yields a sub-Riemannian metric for which the left-translations act as isometries.

The paper is organized as follows.

First, we briefly recall the construction of the cotangent bundle $T^*\mathfrak{h}_{2n+1}$ of the Lie algebra $\mathfrak{h}_{2n+1}$ corresponding to the Heisenberg group H_{2n+1}.

In Section 2 we recall the classification of all non isometric left invariant Riemannian metrics on $T^*\mathfrak{h}_{2n+1}$ (see Theorem 2.3) previously discussed in [12]. Throughout the paper we identify the notion of metric on Lie group and the inner product on its Lie algebra. We obtained only one $n(2n+1)$-parameter family of Riemannian metrics. This family is correlated with the family of Riemannian metrics on Heisenberg group from [14]. Finally, the sub-Riemannian structures are classified in Theorem 2.5.

The Section 3 is devoted to the geodesic curves in both Riemannian and sub-Riemannian setting.

1. Preliminaries

Let us briefly recall the construction of cotangent Lie group.

The *cotangent algebra* $T^*\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ is semidirect product of $\mathfrak{g}$ and its cotangent space $\mathfrak{g}^*$ by means of the coadjoint representation $T^*\mathfrak{g} := \mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$, i.e. the commutators are defined by

$$[(x,\phi),(y,\psi)] := ([x,y], \mathrm{ad}^*(x)(\psi) - \mathrm{ad}^*(y)(\phi)), \quad x,y \in \mathfrak{g}, \ \ \phi,\psi \in \mathfrak{g}^*, \tag{1}$$

where ad^* denotes the coadjoint representation $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g}^*)$ given by $(\mathrm{ad}^*(x)(\phi))(y) := -\phi([x,y])$.

Let us make the following two identifications: identify the zero section of the cotangent bundle of a Lie group G with G itself and the fiber over identity element $(e,0)$ with $\mathfrak{g}^*$. Then, as a Lie group, the *cotangent group* T^*G can be viewed as $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$. The multiplication law is given by:

$$(g,\alpha)\cdot(h,\beta) = (g\cdot h, \mathrm{Ad}^*_{h^{-1}}(\alpha)+\beta), \quad g,h \in G, \ \alpha,\beta \in \mathfrak{g}^*. \tag{2}$$

Now, let us apply this construction to the corresponding Lie algebra $\mathfrak{h}_{2n+1}$ of $(2n+1)$-dimensional Heisenberg group H_{2n+1}.

The standard symplectic form in vector space $\mathbb{R}^{2n}$ can be written in the form $\omega(x,y) = x^T J y$, $x,y \in \mathbb{R}^{2n}$, where the matrix of standard complex structure on $\mathbb{R}^{2n}$ is denoted by

$$J = J_{2n} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \tag{3}$$

and E is the identity matrix of dimension $n \times n$.

The Heisenberg group is two-step nilpotent Lie group H_{2n+1} defined on the base manifold $\mathbb{R}^{2n} \oplus \mathbb{R}$ by multiplication

$$(x,\mu)\cdot(y,\lambda) := (x+y, \mu+\lambda+\omega(x,y)).$$

The corresponding Lie algebra $\mathfrak{h}_{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{R}^{2n} \oplus \mathcal{Z} = \{(x,\mu) \,|\, x \in R^{2n}, \mu \in \mathbb{R}\}$ has the following commutator

$$[(x,\mu),(y,\lambda)] = (0, \omega(x,y)). \tag{4}$$

Note that $\mathcal{Z} = \mathbb{R}\langle\mu\rangle$ is one-dimensional center and one-dimensional commutator subalgebra of $\mathfrak{h}_{2n+1}$. To simplify notation we write (4) in the equivalent form $[x+\mu z, y+\lambda z] = \omega(x,y)z$. If $e_1,\dots,e_n,f_1,\dots,f_n$ represents the standard basis of $\mathbb{R}^{2n}$, then nonzero commutators of $\mathfrak{h}_{2n+1}$ are $[e_i,f_i] = z$, $i = 1,\dots,n$.

Let $\mathbb{R}^{*2n} = \mathbb{R}\langle e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*\rangle$ and $\mathcal{Z}^* = \mathbb{R}\langle z^*\rangle$ be dual vector spaces of one-forms spanned by dual basis vectors. Denote the dual space of $\mathfrak{h}_{2n+1}$ by

$$\mathfrak{h}_{2n+1}^* = \mathbb{R}^{*2n} \oplus \mathcal{Z}^* = \{(x^*, \mu^*) \,|\, x^* \in R^{2n}, \mu^* \in \mathbb{R}\} = \{x^* + \mu^* z^*\}.$$

Now, the cotangent space $\mathrm{T}^*\mathfrak{h}_{2n+1}$ of $\mathfrak{h}_{2n+1}$, as a vector space, can be written as a direct sum

$$\mathfrak{g} = \mathrm{T}^*\mathfrak{h}_{2n+1} = \mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2n+1}^* = \mathbb{R}^{2n} \oplus \mathcal{Z}^* \oplus \mathbb{R}^{*2n} \oplus \mathcal{Z} \cong \mathbb{R}^{4n+2}. \tag{5}$$

Note, that we changed the order of summands to better fit the structure of Lie algebra. Namely, one can check that according to (1) the commutator on $\mathrm{T}^*\mathfrak{h}_{2n+1}$ is given by

$$[(x, \mu^*, x^*, \mu), (y, \lambda^*, y^*, \lambda)] = (0, 0, \lambda^* J^*(x) - \mu^* J^*(y), \omega(x, y)), \tag{6}$$

where $J^* : \mathbb{R}^{2n} \to \mathbb{R}^{*2n}$, $J^*(X) := (JX)^*$ is represented by matrix J given in (3), in the basis $e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*$, i.e. $J^*(e_k) = -f_k^*$, $J^*(f_k) = e_k^*$.

The center and commutator subalgebra of $\mathrm{T}^*\mathfrak{h}_{2n+1}$ both coincide with $\xi = \mathbb{R}^{*2n} \oplus \mathcal{Z}$. By that reason we have changed the order of summands in (5) to group central (and commutator) vectors together. Hence, it is obvious that $\mathrm{T}^*\mathfrak{h}_{2n+1}$ is a Carnot algebra of step two, i.e. a positive graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.

To simplify the notation we write commutator (6) of $\mathrm{T}^*\mathfrak{h}_{2n+1}$ in equivalent form

$$[x + \mu^* z^* + x^* + \mu z, y + \lambda^* z^* + y^* + \lambda z] = \lambda^* J^*(x) - \mu^* J^*(y) + \omega(x, y) z. \tag{7}$$

Finally, the left invariant vector fields of T^*H_{2n+1} are the complete lifts of left invariant vector fields:

$$X_k^c = \frac{\partial}{\partial x_k} - \frac{1}{2} x_{k+n} \frac{\partial}{\partial z} + z^* \frac{\partial}{\partial x_{k+n}^*}, \qquad X_{k+n}^c = \frac{\partial}{\partial x_{k+n}} + \frac{1}{2} x_k \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial x_k^*}, \qquad Z^c = \frac{\partial}{\partial z}, \tag{8}$$

and the vertical lifts of the Maurer-Cartan left invariant forms on H_{2n+1}:

$$X_k^v = \frac{\partial}{\partial x_k^*}, \qquad X_{k+n}^v = \frac{\partial}{\partial x_{k+n}^*}, \qquad Z^v = \frac{\partial}{\partial z^*}, \tag{9}$$

for $k = 1, \dots, n$. To avoid unnecessary complications with labeling, we used the same symbols to denote coordinates on Lie algebra and Lie group.

From (2) one can derive that the multiplication on T^*H_{2n+1} has the form:

$$(x, \mu^*, x^*, \mu) \cdot (y, \lambda^*, y^*, \lambda) = (x + y, \mu^* + \lambda^*, x^* + y^* - \mu^* J y, \mu + \lambda + \frac{1}{2}\omega(x, y)), \tag{10}$$

where $(x, \mu), (y, \lambda) \in H_{2n+1}$ and $(x^*, \mu^*), (y^*, \lambda^*) \in \mathfrak{h}_{2n+1}^*$.

2. Riemannian and sub-Riemannian structures

First, let us recall the classification of the non-isometric left invariant Riemannian metrics on $\mathfrak{g} = \mathrm{T}^*\mathfrak{h}_{2n+1}$ considered in [12].

If $\mathfrak{g}$ is a Lie algebra and $\langle \cdot, \cdot \rangle$ inner product on $\mathfrak{g}$ the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called a *metric Lie algebra*. The structure of metric Lie algebra uniquely defines left invariant Riemannian metric on the corresponding simple connected Lie group G and vice versa.

Since Lie algebra $\mathfrak{g}$ is nilpotent and therefore completely solvable, non-isometric metrics on $\mathfrak{g}$ are the non-isomorphic ones. They are obtained by action of group $\mathrm{Aut}(\mathfrak{g})$ on the space of metrics.

Lemma 2.1 ([12]). *The group of automorphism of algebra* $\mathfrak{g} = \mathrm{T}^*\mathfrak{h}_{2n+1}$ *of the form* (5) *in the standard basis is*

$$\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_{2n+1}) = \left\{ F = \begin{pmatrix} F_1 & 0 \\ F_3 & F_4 \end{pmatrix} \mid F_1, F_4 \in Gl_{2n+1}(\mathbb{R}),\, F_3 \in M_{2n+1}(\mathbb{R}) \right\}, \tag{11}$$

$$F_1 = \begin{pmatrix} \bar{F}_1 & v_1 \\ u_1^T & f_1 \end{pmatrix}, \quad F_4 = \begin{pmatrix} f_1 f_4 \bar{F}_1^{-T} - (Jv_1)(Ju_1)^T & -f_4 \bar{F}_1^{-T} u_1 \\ -f_4 v_1^T \bar{F}_1^{-T} & f_4 \end{pmatrix}, \quad \bar{F}_1^T J \bar{F}_1 = f_4 J, \tag{12}$$

$v_1, u_1 \in \mathbb{R}^{2n}$, $f_1, f_4 \in \mathbb{R}\backslash\{0\}$. *Its dimension is* $\dim \mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_{2n+1}) = 6n^2 + 9n + 3$.

Remark 2.2. *Group of automorphisms of Heisenberg algebra* $\mathfrak{h}_{2n+1}$ *is subgroup of* $\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_{2n+1})$ *and can be described as semidirect product of symplectic group* $Sp(2n, \mathbb{R})$*, subgroup of translations isomorphic to* $\mathbb{R}^{2n}$ *and* 1*-dimensional ideal. For more details, see [14].*

Theorem 2.3 ([12]). *Dimension of the moduli space of Riemannian metrics on Lie algebra* $\mathrm{T}^*\mathfrak{h}_{2n+1}$ *is* $n(2n+1)$*. Every such metric is represented by* $(4n+2) \times (4n+2)$ *block matrix*

$$S = \begin{pmatrix} D(\sigma) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{S} & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \tag{13}$$

$\omega > 0$, $D(\sigma) = \mathrm{diag}(\sigma_1, \dots, \sigma_{n-1}, 1, \sigma_1, \dots, \sigma_{n-1}, 1)$, $\sigma_1 \geq \cdots \geq \sigma_{n-1} \geq 1$, $\bar{S}$ *is symmetric positive definite matrix of dimension* $2n \times 2n$ *satisfying* $\bar{S}_{i(n+i)} = 0$, $i = 1, \dots n$.

Remark 2.4. *Symplectic rotations* $R_\theta : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$*:*

$$e_i \mapsto \cos\theta_i\, e_i - \sin\theta_i\, f_i, \qquad f_i \mapsto \sin\theta_i\, e_i + \cos\theta_i\, f_i, \tag{14}$$

by angles $\theta_1, \dots, \theta_n \in \mathbb{R}$*, represent unique automorphisms preserving* $\mathrm{diag}(D(\sigma), 1)$ *if all* σ_k *in* $D(\sigma) = \mathrm{diag}(\sigma_1, \dots, \sigma_{n-1}, 1, \sigma_1, \dots \sigma_{n-1}, 1)$ *are distinct. If some of them are equal, there exist wider class of automorphism preserving* $\mathrm{diag}(D(\sigma), 1)$ *that further simplifies the matrix* $\bar{S}$ *(for details see [12]).*

Now, we are able to focus on the sub-Riemannian case.

Theorem 2.5. *Any left-invariant sub-Riemannian structure* $(\mathcal{D}, g)$ *is isometric to exactly one of the structures* $(\bar{\mathcal{D}}, g^\sigma)$ *given by:*

$$\begin{cases} \bar{\mathcal{D}}(e) = \mathcal{L}(e_1, \dots, e_n, f_1, \dots, f_n, z^*) \\ g^\sigma = \mathrm{diag}(D(\sigma), 1). \end{cases} \tag{15}$$

Proof. First, for an arbitrary distribution $\mathcal{D}$ let us show that there exists an automorphism $\phi \in Aut(\mathrm{T}^*\mathfrak{h}_{2n+1})$ such that $\phi_*\mathcal{D} = \bar{\mathcal{D}}$, $\bar{\mathcal{D}}(e) = \mathcal{L}(e_1, \dots, e_n, f_1, \dots, f_n, z^*)$.

Let $\mathrm{T}^*\mathfrak{h}_{2n+1} = \nu \oplus \xi$, where $\xi = \mathcal{L}(e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*, z)$ denotes the center of the algebra $\mathrm{T}^*\mathfrak{h}_{2n+1}$ and $\nu = \mathcal{L}(e_1, \dots, e_n, f_1, \dots, f_n, z^*)$ its complement. Then every element of the distribution $\mathcal{D}$ has the form $x + y$, $x \in \nu$, $y \in \xi$. From Lemma 2.1 follows that $\phi \in Aut(\mathrm{T}^*\mathfrak{h}_{2n+1})$ is represented by the matrix F given by (11). Hence

$$\phi(x + y) = F_1 x + (F_3 x + F_4 y), \qquad F_1 x \in \nu, \quad F_3 x + F_4 y \in \xi.$$

Since F_3 is completely arbitrary, it can always be obtained that $\mathcal{D}$ is generated with "non-central" elements.

Now, let us consider the inner product on $\mathrm{T}^*\mathfrak{h}_{2n+1}$ represented by the non-singular symmetric matrix $S \in M_{4n+2}(\mathbb{R})$ and the action $F^T S F$, where F is the matrix representation of $\psi \in Aut(\mathrm{T}^*\mathfrak{h}_{2n+1})$ preserving the distribution $\bar{\mathcal{D}}$. The matrix F must satisfy conditions (12) and $F_3 = 0$. Since we are only interested in the restriction of the inner product on the distribution $\bar{\mathcal{D}}$, the problem is reduced to the action of matrix

$$F_1 = \begin{pmatrix} \bar{F}_1 & v_1 \\ u_1^T & f_1 \end{pmatrix}, \quad \bar{F}_1^T J \bar{F}_1 = f_4 J, \quad v_1, u_1 \in \mathbb{R}^{2n}, \; f_1, f_4 \in \mathbb{R},$$

on the symmetric matrix $S_1 \in M_{2n+1}(\mathbb{R})$. The appropriate choice of v_1, u_1 allows us to obtain the form

$$F_1^T S_1 F_1 = \begin{pmatrix} \bar{S}_1 & 0 \\ 0 & \omega_1 \end{pmatrix}, \quad \bar{S}_1^T = \bar{S}_1, \; \omega_1 > 0.$$

Symplectic matrix $\bar{F}_1$ can diagonalize positive symmetric matrix $\bar{S}_1$ (see [14, 15] for details regarding independent action of $\bar{S}_1$ and f_4) to obtain $\bar{F}_1^T \bar{S}_1 \bar{F}_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \sigma_1, \ldots, \sigma_n)$, with $\sigma_1 \geq \ldots \geq \sigma_n > 0$. Finally, by setting $f_1 = \frac{1}{\sqrt{\omega_1}}$ and $f_4 = \frac{1}{\sigma_n}$, $\bar{F}_1^T \bar{S}_1 \bar{F}_1 = \mathrm{diag}(D(\sigma), 1) = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n-1}, 1, \sigma_1, \ldots, \sigma_{n-1}, 1, 1)$ which is exactly the restriction of the Riemannian metric (13) on the distribution $\bar{\mathcal{D}}$.

Hence, the automorphism $\theta = \psi \circ \phi$ is the isometry between two left invariant sub-Riemannian structures $(\mathcal{D}, g)$ and $(\bar{\mathcal{D}}, g^\sigma)$ on T^*H_{2n+1}: $\theta_* \mathcal{D} = \bar{\mathcal{D}}$ and $g = \theta^* g^\sigma$. □

Remark 2.6. *Note that the sub-Riemannian metric g^σ given by* (15) *is precisely the restriction of the corresponding Riemannian metric* (13) *to the distribution $\bar{\mathcal{D}}$. Hence, we say that the sub-Riemannian structure is* tamed *by the Riemannian one.*

3. Geodesics on $\mathbf{T}^*H_{2n+1}$

Geodesic of an arbitrary left invariant metric on a Lie group G can be seen as a motion of a generalized rigid body with a configuration space G. In Riemannian case, a geodesic passing through a fixed point is determined by an initial geodesic vector (tangent vector), while in sub-Riemannian case it is determined by its initial momentum.

There are two main approaches to the definition of geodesics: geodesics as *shortest curves* based on Maupertuis principle of least action (variational approach) and geodesics as *straightest curves* based on d'Alembert's principle of virtual work (which leads to a geometric description, based on the notion of connection). Although the various definitions of geodesics are equivalent in Riemannian case, their generalizations to the sub-Riemannian setting give rise to the non-equivalent objects. For an extensive overview of different approaches to definitions, generalizations and study of geodesics we refer to [2].

The other distinction between Riemannian and sub-Riemannian geometry is the existence of singular geodesics. In Riemannian case all the geodesics are *normal geodesics*, meaning that they are the solutions of geodesic equation. In sub-Riemannian case for some geometries there exist minimizing geodesics that do not solve the geodesic equation. They are called *singular* or *abnormal geodesics*.

We are considering a 2-step nilpotent Lie group, hence there are many known facts about geodesics in both Riemannian and sub-Riemannian case. In sub-Riemannian case, we can rule out the existence of abnormal geodesics since the underlying distribution is contact (see, e.g. [1, Proposition 4.38]). As observed in Remark 2.6, the Riemannian structure tames the sub-Riemannian one, hence the normal sub-Riemannian geodesics are exactly the $\mathcal{D}$-projection of the Riemannian geodesics (see [4, Proposition 7]).

In the sequel we find geodesics in the Riemannian case with respect to metric from Theorem 2.3 and also sub-Riemannian geodesics with respect to structure from Theorem 2.5.

3.1. Riemannian case

Let $c(t)$ be a curve on the Lie group G and let $\gamma(t) = dL_{c(t)}^{-1} \dot{c}(t)$ be its associated curve on the Lie algebra $\mathfrak{g}$. Then it is well known that $c(t)$ is geodesic if and only if:

$$\dot{\gamma} = \mathrm{ad}_\gamma^* \gamma. \tag{16}$$

Recall that the center ξ of the Lie algebra $T^*\mathfrak{h}_{2n+1}$ is non-degenerate with the respect to the metric (13). Define the mapping $j : \xi \to End(\xi^\perp)$ as:

$$j(u)v = \mathrm{ad}_v^*(u), \quad v \in \xi^\perp,\ u \in \xi.$$

If we write $\gamma(t) = v(t) + u(t)$, $v(t) \in \xi^\perp$ and $u(t) \in \xi$, then (16) becomes:

$$\dot{v}(t) = j(u)v, \quad \dot{u}(0) = 0. \tag{17}$$

With the initial conditions $v(0) = v_0$, $u(0) = u_0$ and $j(u_0) = K$, the solution of (17) is:

$$v(t) = \exp(tK)v_0, \quad u(t) = u_0. \tag{18}$$

Obviously, the associated geodesic curves are constant on the center ξ. Furthermore, the matrix K is given by:

$$K = \begin{pmatrix} -\omega\mu_0 D^{-1}(\sigma)J & D^{-1}(\sigma)J\bar{S}x_0^* \\ (x_0^*)^T\bar{S}J & 0 \end{pmatrix}, \tag{19}$$

where $u_0 = (x_0^*, \mu_0)$. Note that the matrix K is skew-symmetric if and only if the matrix $D(\sigma)$ is an identity matrix (this is always true for $n = 1$). In that case the associated curves on $\xi^\perp$ are circles.

Example 3.1. *Let us thoroughly investigate the case when $n = 1$. The metric* (13) *takes the diagonal form $S = diag(1, 1, 1, s_1, s_2, \omega)$. Let $\gamma(t) = x(t) + y(t) + \mu^*(t) + x^*(t) + y^*(t) + \mu(t)$, then the equation* (16) *becomes the following:*

$$\begin{aligned} \dot{x} &= -s_2 y^*\mu^* + \omega\mu y, \\ \dot{y} &= \ \ s_1 x^*\mu^* - \omega\mu x, \\ \dot{\mu^*} &= \ \ s_2 y^* x - s_1 x^* y, \\ \dot{x^*} &= \dot{y^*} = \dot{\mu} = 0. \end{aligned}$$

Since, $x^*(t) = x^*(0) = x_0^*$, $y^*(t) = y^*(0) = y_0^*$ and $\mu(t) = \mu(0) = \mu_0$, the previous system can be written in simpler matrix notation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu^*} \end{pmatrix} = \begin{pmatrix} 0 & \omega\mu_0 & -s_2 y_0^* \\ -\omega\mu_0 & 0 & s_1 x_0^* \\ s_2 y_0^* & -s_1 x_0^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu^* \end{pmatrix}.$$

Put $\tau^2 = (s_1 x_0^*)^2 + (s_2 y_0^*)^2 + (\omega\mu_0)^2$ and $\varsigma = s_1 x_0 x_0^* + s_2 y_0 y_0^* + \omega\mu_0\mu_0^*$. If $\tau > 0$, then the solution of the previous system is:

$$\begin{aligned} x(t) &= \frac{s_1 x_0^*\varsigma}{\tau^2} + \frac{-s_2 y_0^*\mu_0^* + \omega y_0\mu_0}{\tau}\sin(\tau t) + \frac{x_0\tau^2 - s_1 x_0^*\varsigma}{\tau^2}\cos(\tau y), \\ y(t) &= \frac{s_2 y_0^*\varsigma}{\tau^2} + \frac{s_1 x_0^*\mu_0^* - \omega x_0\mu_0}{\tau}\sin(\tau t) + \frac{y_0\tau^2 - s_2 y_0^*\varsigma}{\tau^2}\cos(\tau t), \\ \mu_0^*(t) &= \frac{\omega\mu_0\varsigma}{\tau^2} + \frac{-s_1 x_0^* y_0 + s_2 x_0 y_0^*}{\tau}\sin(\tau t) + \frac{\mu_0^*\tau^2 - \omega\mu_0\varsigma}{\tau^2}\cos(\tau t), \end{aligned}$$

where $x(0) = x_0$, $y(0) = y_0$ and $\mu^*(0) = \mu_0^*$. If $\tau = 0$, then the solution is simply:

$$x(t) = x_0, \quad y(t) = y_0, \quad \mu^*(t) = \mu_0^*.$$

The geodesic curve $c(t)$ on Lie group can be retrieved from the previous equations by integration.

3.2. Sub-Riemannian case

The minimizing geodesic is an absolutely continuous horizontal curve that realizes the Carnot-Carathéodory distance:

$$d(A,B) = \inf\{\ell(\gamma) = \int \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle} dt \mid \gamma - \text{horizontal curve connecting points } A \text{ and } B\}, \tag{20}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the distribution $\mathcal{D}$ and the integral is over the domain of the curve.

The existence of minimizing sub-Riemannian geodesics is a consequence of the Chow's theorem: on a bracket-generating sub-Riemannian manifold G any two sufficiently close points can be joined by a minimizing geodesic. Additionally, if G is connected and complete relative to the sub-Riemannian distance function (20), then any two points can be joined by a minimizing geodesic (see [10, Theorem 1.6.3 and Theorem 1.6.4] for the proof of local and global existence of minimizing geodesics, respectively).

To simplify the notation, in the following we will use the local coordinates (x, z), where z are $2n+1$ central coordinates. In this coordinates the vector fields (8), (9) have the form:

$$X_k = \frac{\partial}{\partial x_k} - \frac{1}{2}\sum_{j,i=1}^{2n+1} c^i_{kj} x_j \frac{\partial}{\partial z_i}, \quad k = 1, \dots, 2n+1, \qquad Z_i = \frac{\partial}{\partial z_i}, \quad i = 1, \dots 2n+1,$$

where X_k span $\mathcal{D}$ and $C_i = (c^i_{kj})$, $i = 1, \dots, 2n+1$, are skew-symmetric matrices (the coefficients c^i_{kj} can be obtained from structural equations (6)).

Now, we can introduce a system of coordinates (h, w) on fibers of $\mathrm{T}^*(\mathrm{T}^* H_{2n+1})$:

$$h_k(\lambda) = \langle \lambda, X_k(g)\rangle, \quad w_i(\lambda) = \langle \lambda, Z_i(g)\rangle, \quad k, i = 1, \dots, 2n+1,$$

with $\langle \cdot, \cdot \rangle$ used to identify vectors and covectors.

The sub-Riemannian Hamiltonian (or kinetic energy) is the fiber-quadratic function H. The sub-Riemannian structure is uniquely determined by its Hamiltonian (see [10, Proposition 1.5.3]). The normal extremal trajectories $\lambda(t) = (x(t), z(t), h(t), w(t))$ are projections of integral curves of $H = \frac{1}{2}\sum_{k=1}^{2n+1} \frac{1}{\sigma_k} h_k^2$, with $\sigma_{2n+1} = 1$. In coordinate terms, the normal geodesic equations can be written in the form:

$$\begin{aligned} \dot x_k &= \{x_k, H\} = \frac{1}{\sigma_k} h_k, & \dot h_k &= \{H, h_k\} = -\sum_{j=1}^{2n+1} \frac{1}{\sigma_j}\{h_k, h_j\} h_j = -\sum_{j,i=1}^{2n+1} \frac{1}{\sigma_j} c^i_{kj} h_j w_i, \\ \dot z_i &= \{z_i, H\} = -\frac{1}{2}\sum_{j,k=1}^{2n+1} \frac{1}{\sigma_k} c^i_{kj} h_k x_j, & \dot w_i &= \{H, w_i\} = 0, \end{aligned} \tag{21}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The solution of Hamiltonian system of differential equations is:

$$h(t) = \exp(-t\Omega_w) h(0), \quad \Omega_w = \sum_{i=1}^{2n+1} w_i C_i = \begin{pmatrix} w_{2n+1} D^{-1}(\sigma) J & -J\bar w \\ -\bar w^T J & 0 \end{pmatrix},$$

$$x(t) = x(0) + \int_0^t \exp(-s\Omega_w) h(0)\, ds,$$

where vector $w = (\bar w, w_{2n+1})$ is constant and vertical coordinates z can always be recovered.

Example 3.2. *Consider the case $n = 1$. The system* (21) *has the following form:*

$$\begin{aligned} \dot x_k &= h_k, \quad k = 1, 2, 3, & \dot w_k &= 0, \quad k = 1, 2, 3, \\ \dot h_1 &= w_3 h_2 - w_2 h_3, & \dot z_1 &= \frac{1}{2}(h_2 x_3 - h_3 x_2), \\ \dot h_2 &= -w_3 h_1 + w_1 h_3, & \dot z_2 &= \frac{1}{2}(h_3 x_1 - h_1 x_3), \\ \dot h_3 &= w_2 h_1 - w_1 h_2, & \dot z_3 &= \frac{1}{2}(h_2 x_1 - h_1 x_2). \end{aligned}$$

The solution of this system is:

$$
\begin{aligned}
x_1(t) &= x_1(0) + \frac{w_1\varsigma}{\tau^2}t + \frac{w_3h_2(0) - w_2h_3(0)}{\tau^2}\cos(\tau t) + \frac{\tau^2h_1(0) - \varsigma w_1}{\tau^3}\sin(\tau t),\\
x_2(t) &= x_2(0) + \frac{w_2\varsigma}{\tau^2}t + \frac{w_3h_1(0) - w_1h_3(0)}{\tau^2}\cos(\tau t) + \frac{\tau^2h_2(0) - \varsigma w_2}{\tau^3}\sin(\tau t),\\
x_3(t) &= x_3(0) + \frac{w_3\varsigma}{\tau^2}t + \frac{w_2h_1(0) - w_1h_2(0)}{\tau^2}\cos(\tau t) + \frac{\tau^2h_3(0) - \varsigma w_3}{\tau^3}\sin(\tau t),\\
h_1(t) &= \frac{w_1\varsigma}{\tau^2} + \frac{\tau^2h_1(0) - \varsigma w_1}{\tau^2}\cos(\tau t) - \frac{w_3h_2(0) - w_2h_3(0)}{\tau}\sin(\tau t),\\
h_2(t) &= \frac{w_2\varsigma}{\tau^2} + \frac{\tau^2h_2(0) - \varsigma w_2}{\tau^2}\cos(\tau t) - \frac{w_3h_1(0) - w_1h_3(0)}{\tau}\sin(\tau t),\\
h_3(t) &= \frac{w_3\varsigma}{\tau^2} + \frac{\tau^2h_3(0) - \varsigma w_3}{\tau^2}\cos(\tau t) - \frac{w_2h_1(0) - w_1h_2(0)}{\tau}\sin(\tau t),\\
w_1(t) &= w_1, \quad w_2(t) = w_2, \quad w_3(t) = w_3,
\end{aligned}
$$

with $\tau^2 = w_1^2 + w_2^2 + w_3^2$ *and* $\varsigma = w_1h_1(0) + w_2h_2(0) + w_3h_3(0)$. *The expression for z component of the solution is too complicated to be written, but it can easily be obtained. Obviously, if* $\tau = 0$*, the solution is:*

$$
\begin{aligned}
&x_k(t) = x_k(0) + h_kt, \quad h_k(t) = h_k(0), \quad w_k(t) = 0, \quad \textit{for } k = 1, 2, 3,\\
&z_1(t) = z_1(0) + \frac{1}{2}(h_2(0)x_3(0) - h_3(0)x_2(0))t,\\
&z_2(t) = z_2(0) + \frac{1}{2}(h_3(0)x_1(0) - h_1(0)x_3(0))t,\\
&z_3(t) = z_3(0) + \frac{1}{2}(h_2(0)x_1(0) - h_1(0)x_2(0))t.
\end{aligned}
$$

Remark 3.3. *Note that process of obtaining the associated curves* (18) *works for all 2-step nilpotent groups (see [7]), while the sub-Riemannian case is valid for all 2-step Carnot groups (see [1]). Note that in general sub-Riemannian geodesic flows on Carnot groups are not integrable (see [5]).*

References

[1] A.A. Agrachev, D. Barilari, U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, Cambridge University Press, 2019.

[2] D. V. Alekseevskiĭ, *Shortest and straightest geodesics in sub-Riemannian geometry*, J. Geom. Phys. 2020; 155: 21pp.

[3] R. Biggs, P. T. Nagy, *A classification of sub-Riemannian structures on the Heisenberg groups*, Acta Polytech. Hungar. 2013; 10(7): 41–52.

[4] R. Biggs, P. T. Nagy, *On sub-Riemannian and Riemannian structures on the Heisenberg groups*, J. Dyn. Control. Syst. 2016; 22(3): 563–594.

[5] I. A. Bizyaev, A. V. Borisov, A. A. Kilin, I. S. Mamaev, *Integrability and nonintegrability of sub-Riemannian geodesic flows on Carnot groups*, RRegul. Chaotic Dyn. 2016; 21: 759–774.

[6] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. de l' Éc. Norm. 1994; 27(5): 611–660.

[7] M. Guediri, *Sur la complétude des pseudo-métriques invariantes à gauche sur les groupes de Lie nilpotents,* Rend. Sem. Mat. Univ. Pol. Torino 52 (1994), 371–376.

[8] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull London Math Soc. 1983; 15(1): 35–42.

[9] J. Lauret, *Homogeneous nilmanifolds of dimension* 3 *and* 4, Geom. Dedicata. 1997; 68: 145–155.

[10] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics, and Applications*, American Mathematical Soc., 2002.

[11] Yu. L. Sachkov, *Left-invariant optimal control problems on Lie groups: classification and problems integrable by elementary functions*, Russian Math. Surv. 2022; 77: 99—163.

[12] T. Šukilović, S. Vukmirović, *Geometry of cotangent bundle of Heisenberg group*, Diff. Geom. Appl. 88 (2023): 101997.

[13] A.M. Vershik, V.Y. Gershkovich, *Nonholonomic problems and the theory of distributions*, Acta Appl. Math. 1988; 12(2): 181–209.

[14] S. Vukmirović, *Classification of left-invariant metrics on the Heisenberg group*, J. Geom. Phys. 2015; 94: 72–80.

[15] J. Williamson, *On the algebraic problem concerning the normal forms of linear dynamical systems*, Am. J. Math. 58, 141 (1936).