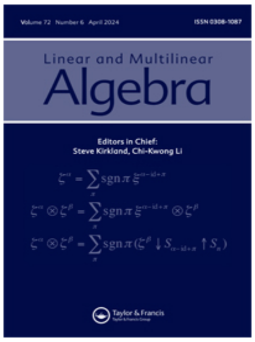

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Norm inequalities for hyperaccretive quasinormal operators, with extensions of the arithmetic-geometric means inequality

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ABSTRACT

Let Φ, Ψ be symmetrically norming (s.n.) functions, let $\mathfrak{C}_\Psi(\mathcal{H})$ be the ideal of compact Hilbert space operators, associated with the s.n. function Ψ, $p \geqslant 2$ and let $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that A, B are accretive and $AX + XB \in \mathfrak{C}_\Psi(\mathcal{H})$. Then $\sqrt{A^* + A}X\sqrt{B + B^*} \in \mathfrak{C}_\Psi(\mathcal{H})$ as well, and

$$\left\| \sqrt{A^* + A}X\sqrt{B + B^*} \right\|_\Psi \leqslant \|AX + XB\|_\Psi,$$

under any of the following conditions:

(a) $\Psi := \Phi^{(p)^*}$ and A (resp. B^*) is quasinormal operator with its adjoint operator being 2-hyperaccretive and having the injective real part;

(b) if both A and B^* are quasinormal operators with its adjoint operators being 2-hyperaccretive operators and having injective real parts.

Also, for $M, N \in \mathbb{N}$ which are such that $K := (M + N)/2 \in \mathbb{N}$, for $I, J \in \mathbb{N} \cup \{0\}$, for s.n. functions Φ, Ψ, for bounded Hilbert space operators A, B, X, such that A is $(M + I)$-hyperaccretive and B^* is $(N + J)$-hyperaccretive, satisfying $\Delta_{A,B}^K(X) \stackrel{\text{def}}{=} \sum_{k=0}^K \binom{K}{k} A^{K-k}XB^k \in \mathfrak{C}_\Psi(\mathcal{H})$, then there exists $V_{A,B}^{M+I,N+J,-}X \stackrel{\text{def}}{=} {}^w\lim_{T\to+\infty} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{1/2} \mathrm{e}^{-TA}X\,\mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{1/2}$ and

$$\left\| \left(\sum_{m=0}^{M+I} \binom{M+I}{m} A^{*m}A^{M+I-m} \right)^{\frac{1}{2}} X \left(\sum_{n=0}^{N+J} \binom{N+J}{n} B^{N+J-n}B^{*n} \right)^{\frac{1}{2}} - V_{A,B}^{M+I,N+J,-}X \right\|_\Psi$$

$$\leqslant \frac{\sqrt{(M-1)!(N-1)!}}{\left(\frac{M+N}{2} - 1\right)!} \left\| \left(\sum_{i=0}^{I} \binom{I}{i} A^{*i}A^{I-i} \right)^{\frac{1}{2}}\right.$$

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$$\left.\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}B^{*j}\right)^{\frac{1}{2}}\right\|_{\Psi},$$

hold under any of the following conditions:

(a) if $p \geqslant 2$, $\Psi := \Phi^{(p)*}$ and A or B^* is normal (in which case $V_{A,B}^{M+I,N+J,-}X = 0$),
(b) if both A and B^* are normal (in which case $V_{A,B}^{M+I,N+J,-}X = 0$),
(c) if $\|\cdot\|_\Psi := \|\cdot\|_1$ and $AX + XB \in \mathcal{C}_1(\mathcal{H})$.

1. Introduction

Let $\mathcal{H}$ be separable, complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ denote the spaces of all bounded and all compact linear operators, respectively. Each 'symmetric gauge' or 'symmetrically norming' (s.n.) function Φ, defined on sequences of complex numbers, gives rise to a unitary invariant (u.i.) norm $\|\cdot\|_\Phi$ on operators. Basic examples of s.n. functions are the trace s.n. function ℓ (also denoted by ℓ^1 or ℓ_1), defined by $\ell\left((\lambda_n)_{n=1}^\infty\right) \stackrel{\text{def}}{=} \sum_{n=1}^\infty |\lambda_n|$ and the ($\mathcal{B}(\mathcal{H})$ or) operator norm ℓ^∞ defined by $\ell^\infty\left((\lambda_n)_{n=1}^\infty\right) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} |\lambda_n|$. Any such norm is unitarily invariant (u.i.) and it is defined on the naturally associated norm ideal $\mathcal{C}_\Phi(\mathcal{H})$ of $\mathcal{C}_\infty(\mathcal{H})$. For a s.n. function Φ, its adjoint s.n. function will be denoted by Φ^*. Each $\|\cdot\|_\Phi$ is lower semicontinuous, i.e. $\|\,{}^w\!\lim_{n\to\infty} X_n\|_\Phi \leqslant \liminf_{n\to\infty} \|X_n\|_\Phi$, where ${}^w\!\lim_{n\to\infty}$ (resp. ${}^s\!\lim_{n\to\infty}$) denotes the weak (resp. strong) operator limit. This follows from the well-known formula $\|X\|_\Phi = \sup\left\{\frac{|\mathrm{tr}(XY)|}{\|Y\|_{\Phi^*}} : Y is finite rank operator\right\}$. For any $p > 0$ a s.n. function Φ could be p modified and its modification $\Phi^{(p)}$ represents a new s.n. function (only) for $p \geqslant 1$. Also, the corresponding ideals of compact operators will be denoted by $\mathcal{C}_{\Phi^{(p)}}(\mathcal{H})$ and their duals by $\mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$, while the closure of finite rank operators in the $\|\cdot\|_{\Phi^{(p)}}$ will be denoted by $\mathcal{C}^{(\circ)}_{\Phi^{(p)}}(\mathcal{H})$. The Schatten-von Neumann trace classes $\mathcal{C}_p(\mathcal{H}) \stackrel{\text{def}}{=} \mathcal{C}_{\ell^{(p)}}(\mathcal{H})$ represent classic examples of norm ideals associated with degree p modified (i.e. its s.n. function ℓ) norms. $\mathcal{C}_1(\mathcal{H})$ is also known as the class of nuclear operators, while $\mathcal{C}_2(\mathcal{H})$ is known as the Hilbert–Schmidt class. The norm in $\mathcal{C}_p(\mathcal{H})$ will be denoted simply by $\|\cdot\|_p$. For $p \geqslant 2$, all norms $\|\cdot\|_{\Phi^{(p)}}$ are also known as Q-norms, as $\Phi^{(p)} = \left(\Phi^{(\frac{p}{2})}\right)^{(2)}$ and $\Phi^{(\frac{p}{2})}$ is also a s.n. function, while its dual norms $\|\cdot\|_{\Phi^{(p)*}}$ are commonly known as Q*-norms. Moreover, we have the following monotonicity property for u.i. norms, saying that

$$\|AXB\|_\Phi \leqslant \|CXD\|_\Phi, \tag{1}$$

whenever $A^*A \leqslant C^*C$ and $BB^* \leqslant DD^*$. For the proof, see [1, p. 62] and note that (1) immediately implies $\|AXB\|_\Phi = \||A|X|B^*|\|_\Phi$. In the normed space $(\mathcal{X}, \|\cdot\|)$ and a sequence $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ we write $\boldsymbol{x} = {}^{\mathcal{X}}\lim_{n\to\infty} \boldsymbol{x}_n$ for some $\boldsymbol{x} \in \mathcal{X}$ if $\lim_{n\to\infty} \|\boldsymbol{x}_n - \boldsymbol{x}\| = 0$. Also, $\boldsymbol{x} = {}^{\mathcal{X}}\sum_{n=1}^{\infty} \boldsymbol{x}_n$ will denote that $\boldsymbol{x} = {}^{\mathcal{X}}\lim_{n\to\infty} \sum_{k=1}^{n} \boldsymbol{x}_k$. If $\mathcal{X} := \mathfrak{C}_\Psi(\mathcal{H})$ for some s.n. function Ψ, through this paper, we use abbreviation ${}^{\Psi}\lim := {}^{\mathfrak{C}_\Psi(\mathcal{H})}\lim$. Exceptions from this notation will be the strong (resp. weak) operator limit and sum ${}^{s}\lim_{n\to\infty}$ and ${}^{s}\sum_{n=1}^{\infty}$ (resp. ${}^{w}\lim_{n\to\infty}$ and ${}^{w}\sum_{n=1}^{\infty}$).

If $(\Omega, \mathfrak{M}, \mu)$ is a space Ω with a measure μ on the σ-algebra $\mathfrak{M}$, then we will refer to a function $A\colon \Omega \to \mathcal{B}(\mathcal{H})\colon t \mapsto A_t$ as a weakly*-measurable if $t \mapsto \langle A_t g, h\rangle$ is measurable for all $g, h \in \mathcal{H}$. If, in addition, those functions are integrable, then there is the unique (known as Gel'fand or weak*-integral) and denoted by $\int_\Omega A_t \,\mathrm{d}\mu(t))$ operator in $\mathcal{B}(\mathcal{H})$, satisfying $\left\langle \int_\Omega A_t \,\mathrm{d}\mu(t) h, k\right\rangle = \int_\Omega \langle A_t h, k\rangle \,\mathrm{d}\mu(t)$ for all $h, k \in \mathcal{H}$. Thus, it also complies with the definition of a Pettis integral. For a more complete account of weak*-integrals, the reader is referred to [2, p.53], [3, p.320] and [4, Lemma 1.2]. For every $h \in \mathcal{H}$, the function $t \mapsto \|A_t h\|$ is also measurable, and, if additionally $\int_\Omega \|A_t h\|^2 \,\mathrm{d}\mu(t) < +\infty$ for all $h \in \mathcal{H}$, then there exists a weak*-integral $\int_\Omega A_t^* A_t \,\mathrm{d}\mu(t) \in \mathcal{B}(\mathcal{H})$, satisfying $\left\langle \int_\Omega A_t^* A_t \,\mathrm{d}\mu(t)\, h, h\right\rangle = \int_\Omega \|A_t h\|^2 \,\mathrm{d}\mu(t)$ for all $h \in \mathcal{H}$, as shown in [3, Ex.2]. Such families $\{A_t\}_{t\in\Omega}$ will be called $[\mu]$ square integrable on Ω ($[\mu]$ s.i. or simply s.i.). If a family $\{A_t\}_{t\in\Omega}$ or $\{A_n\}_{n=1}^{\infty}$ consists of mutually commuting normal operators, we will refer to it as to a m.c.n.o. family. The terminology used in this paper is closely related to that used in Refs [1,5], and a more detailed introduction therein may contribute to the comfort of the reader of this article. For a more complete account of the theory of norm ideals, the interested reader is referred to Refs [6–8].

In this paper we present some extensions of the arithmetic-geometric (A-G) means inequalities for N-hyperaccretive operators. If $A, B \in \mathcal{B}(\mathcal{H})$ then $A \otimes B$ will stand for the bilateral multiplier $A \otimes B\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\colon X \mapsto AXB$. For accretive $A, B \in \mathcal{B}(\mathcal{H})$ and contractive operators $C, D \in \mathcal{B}(\mathcal{H})$, some aspects of the accretive derivations $\Delta_{A,B} \stackrel{\text{def}}{=} A\otimes I + I\otimes B$ and contractive perturbations (of the identity transformer) $\Gamma_{C,D} \stackrel{\text{def}}{=} I\otimes I - C\otimes D$ on various norm ideals of compact operators will also be investigated. Let us recall the Newton-Leibnitz formula (9) in Ref. [9]

$$X - \mathrm{e}^{-TA} X \,\mathrm{e}^{-TB} = \int_{[0,T]} \mathrm{e}^{-tA} (AX + XB)\, \mathrm{e}^{-tB} \,\mathrm{d}t \quad \text{for all } A, B, X \in \mathcal{B}(\mathcal{H}), T > 0. \tag{2}$$

Throughout this paper, we use the notation $\mathbb{N}_\circ \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ \stackrel{\text{def}}{=} [0, +\infty)$.

We also need to emphasize that throughout this paper, we will treat (address to) every unnumbered line in a multiline formula as a part of the consequent numbered one.

2. Preliminaries

First let us recall the next simple convergence principles for u.i. norms and a multiplication property, including $\mathcal{B}(\mathfrak{C}_\Psi(\mathcal{H}))$ (bounded transformers) valued functions.

Lemma 2.1: *Let Ψ be s.n. function, $A, B, C, V\colon \mathbb{R} \to \mathcal{B}(\mathcal{H})$ and $X\colon \mathbb{R} \to \mathfrak{C}_\Psi(\mathcal{H})$.*

(a) *If $A_\circ \stackrel{\text{def}}{=} {}^{\mathcal{B}(\mathcal{H})}\lim_{\mathbb{R}\ni t\to t_\circ} A_t$, $B_\circ \stackrel{\text{def}}{=} {}^{\mathcal{B}(\mathcal{H})}\lim_{\mathbb{R}\ni t\to t_\circ} B_t$ and $X_\circ \stackrel{\text{def}}{=} {}^{\Psi}\lim_{\mathbb{R}\ni t\to t_\circ} X_t$, then*

$$\mathop{{}^{\Psi}\lim}_{\mathbb{R}\ni t\to t_\circ} A_t X_t B_t = A_\circ X_\circ B_\circ. \tag{3}$$

(b) *If* ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}\begin{bmatrix} A_t & B_t \\ 0 & C_t\end{bmatrix}=0$, *then* ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ} A_t=0$.

(c) *If additionally* $A_t\geqslant 0$ *for all* $t\in\mathbb{R}$, *then* ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ} A_t=0$ *if and only if* ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}\sqrt{A_t}=0$. *Moreover, under any of those equivalent conditions,* ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ} V_t\sqrt{A_t}=0$ *if* $\sup\|V_t\|<+\infty$, *where the supremum is taken for t in some neighbourhood of* $t_\circ$.

(d) *If* ${}^{s}\lim_{t\to t_\circ} A_t=0$ *for a family* $\{A_t\}_{t\in\mathbb{R}}$ *of normal operators in* $\mathcal{B}(\mathcal{H})$ *and* ${}^{w}\lim_{t\to t_\circ} B_t=B_\circ$ *for a family* $\{B_t\}_{t\in\mathbb{R}}$ *in* $\mathcal{B}(\mathcal{H})$, *then* ${}^{w}\lim_{t\to t_\circ} A_tB_t=0$.

(e) *If* $\mathcal{T},\mathcal{S}\colon\mathbb{R}_+\to\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))$ *and* $\lim_{t\searrow 0}\big(\|\mathcal{T}_t-\mathcal{T}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}+\|\mathcal{S}_t-\mathcal{S}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}\big)$ $=0$, *then*

$$\lim_{t\searrow 0}\|\mathcal{T}_t\mathcal{S}_t-\mathcal{T}_0\mathcal{S}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}=0, \tag{4}$$

$$ {}^{\Psi}\lim_{t\searrow 0}\mathcal{T}_t\mathcal{S}_t(X)=\mathcal{T}_0\mathcal{S}_0(X)\quad\text{for all } X\in\mathcal{C}_\Psi(\mathcal{H}). \tag{5}$$

Proof: It suffices to prove the case $B_t\equiv I$ for the formula (3) in the case (a), which follows from the estimate $\|A_tX_t-A_\circ X_\circ\|_\Psi=\|(A_t-A_\circ)(X_t-X_\circ)+(A_t-A_\circ)X_\circ+A_\circ(X_t-X_\circ)\|_\Psi\leqslant\|A_t-A_\circ\|\|X_t-X_\circ\|_\Psi+\|A_t-A\|\|X_\circ\|_\Psi+\|A_\circ\|\|X_t-X_\circ\|_\Psi\to 0$ as $t\to t_\circ$.

The condition ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}\begin{bmatrix} A_t & B_t \\ 0 & C_t\end{bmatrix}=0$ in (b) implies that $\lim_{t\to t_\circ}\left\|\begin{bmatrix} A_th \\ 0\end{bmatrix}\right\|=\lim_{t\to t_\circ}\left\|\begin{bmatrix} A_t & B_t \\ 0 & C_t\end{bmatrix}\begin{bmatrix} h \\ 0\end{bmatrix}\right\|=0$ for all $h\in\mathcal{H}$, so ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}A_t=0$, as proclaimed.

The condition ${}^{s}\lim\limits_{\mathbb{R}\ni t\to t_\circ}A_t=0$ in (c) implies $\lim\limits_{\mathbb{R}\ni t\to t_\circ}\|\sqrt{A_t}h\|^2=\lim\limits_{\mathbb{R}\ni t\to t_\circ}\langle A_th,h\rangle\leqslant\lim\limits_{\mathbb{R}\ni t\to t_\circ}\|A_th\|\|h\|=0$ for all $h\in\mathcal{H}$, so ${}^{s}\lim\limits_{\mathbb{R}\ni t\to t_\circ}\sqrt{A_t}=0$. In the other direction, the condition ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}\sqrt{A_t}=0$ implies ${}^{s}\lim_{n\to\infty}\sqrt{A_{t_n}}=0$ and therefore $M:=\sup_{n\in\mathbb{N}}\|\sqrt{A_{t_n}}\|<+\infty$ for any sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}\setminus\{t_\circ\}$, according to the uniform boundedness principle. Thus $\lim_{n\to\infty}\|A_{t_n}h\|\leqslant M\lim_{n\to\infty}\|\sqrt{A_{t_n}}h\|=0$ for all $h\in\mathcal{H}$, implying ${}^{s}\lim_{n\to\infty}A_{t_n}=0$ for any sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}\setminus\{t_\circ\}$ such that $\lim_{n\to+\infty}t_n=t_\circ$, which shows that ${}^{s}\lim_{\mathbb{R}\ni t\to t_\circ}A_t=0$. The proof for (c) completes due to the estimate $\lim_{\mathbb{R}\ni t\to t_\circ}\left\|V_t\sqrt{A_t}h\right\|\leqslant\sup\|V_t\|\lim_{\mathbb{R}\ni t\to t_\circ}\left\|\sqrt{A_t}h\right\|=0$ for all $h\in\mathcal{H}$.

In the case (d), we note that $\lim_{t\to t_\circ}\|A_t^*h\|=\lim_{t\to t_\circ}\|A_th\|=0$ for all $h\in\mathcal{H}$, so ${}^{s}\lim_{t\to t_\circ}A_t^*=0$, which combined with the fact that ${}^{w}\lim_{t\to t_\circ}B_t^*=B_\circ^*$ implies ${}^{w}\lim_{t\to t_\circ}B_t^*A_t^*=B_\circ^*\cdot 0=0$. Consequently, ${}^{w}\lim_{t\to t_\circ}A_tB_t={}^{w}\lim_{t\to t_\circ}(B_t^*A_t^*)^*=\big({}^{w}\lim_{t\to t_\circ}(B_t^*A_t^*)\big)^*=0^*=0$, as proclaimed.

The formula (4) in (e) follows from the estimate

$$\begin{aligned}\|\mathcal{T}_t\mathcal{S}_t-\mathcal{T}_0\mathcal{S}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}&=\|(\mathcal{T}_t-\mathcal{T}_0)\,\mathcal{S}_0+(\mathcal{T}_0+(\mathcal{T}_t-\mathcal{T}_0))\,(\mathcal{S}_t-\mathcal{S}_0)\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}\\&\leqslant\|\mathcal{T}_t-\mathcal{T}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}\|\mathcal{S}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}+\big(\|\mathcal{T}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}+\|\mathcal{T}_t-\mathcal{T}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}\big)\\&\times\|\mathcal{S}_t-\mathcal{S}_0\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))},\end{aligned}$$

which tends 0 as $t\searrow 0$, while (5) is a simple consequence of the formula (4). ■

Also, let us remind the following.

Definition 2.2: If $M, N \in \mathbb{N}$ and $A, B, C, Q, S, T, X \in \mathcal{B}(\mathcal{H})$, then we say that:

1° A is (**strictly**) **accretive** if ($A^* + A \geqslant cI$ for some $c > 0$) $A^* + A \geqslant 0$;
2° A is N-**accretive** if $\Delta^N_{A^*,A}(I) = \sum_{n=0}^{N} \binom{N}{n} A^{*n}A^{N-n} \geqslant 0$;
3° A is N-**hyperaccretive** if $\Delta^K_{A^*,A}(I) = \sum_{k=0}^{K} \binom{K}{k} A^{*k}A^{K-k} \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$,
4° C is (**strictly**) **contractive** if ($\|C\| < 1$) $\|C\| \leqslant 1$;
5° C is N-**hypercontractive** if $\Gamma^K_{C^*,C}(I) = \sum_{k=0}^{K}(-1)^k\binom{K}{k}C^{*k}C^k \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$,
6° Q is **quasinormal** if Q and Q^*Q commute, i.e. iff $QQ^*Q = Q^*Q^2$,
7° S is **subnormal** if there is a Hilbert space $\mathcal{K}$ and operators $V \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $W \in \mathcal{B}(\mathcal{K})$, such that $\left[\begin{smallmatrix} S & V \\ 0 & W \end{smallmatrix}\right]$ represents a normal operator on $\mathcal{H} \oplus \mathcal{K}$,
8° T is **hyponormal** if $TT^* \leqslant T^*T$,
9° $A_C \stackrel{\text{def}}{=} \mathcal{A}(C) \stackrel{\text{def}}{=} {}^s\!\lim_{n\to\infty} C^{*n}C^n$ for every contractive $C \in \mathcal{B}(\mathcal{H})$,
10° $\mho^-_{A,B}(X) \stackrel{\text{def}}{=} {}^w\!\lim_{t\to+\infty} \mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}$ (whenever it exists),
11° $V^{M,N,-}_{A,B}X \stackrel{\text{def}}{=} {}^w\!\lim_{T\to+\infty}\sqrt{\Delta^M_{A^*,A}(I)}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{\Delta^N_{B,B^*}(I)}$ (whenever it exists), and especially $V^-_{A,B}X \stackrel{\text{def}}{=} V^{1,1,-}_{A,B}X = {}^w\!\lim_{T\to+\infty}\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{B+B^*}$.

It follows from the very definition 6° in 2.2 that $Q^{*n}Q^n = (Q^*Q)^n$ for all quasinormal operators Q and $n \in \mathbb{N}$. Also, quasinormal operators are subnormal, as $\left[\begin{smallmatrix} Q & \sqrt{Q^*Q-QQ^*} \\ 0 & Q^* \end{smallmatrix}\right]$ is a normal extension of Q from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$.

Example 2.3: If $C \in \mathcal{B}(\mathcal{H})$ is quasinormal contraction, then

$$\Gamma^N_{C^*,C}(I) = \sum_{n=0}^{N}(-1)^n\binom{N}{n}C^{*n}C^n = \sum_{n=0}^{N}(-1)^n\binom{N}{n}(C^*C)^n = (I - C^*C)^N \geqslant 0,$$

so C is N-hypercontractive for all $N \in \mathbb{N}$. Similarly, if $A \in \mathcal{B}(\mathcal{H})$ normal accretive operator, then $\Delta^N_{A^*,A}(I) = (A^* + A)^N \geqslant 0$, so A is N-hyperaccretive for all $N \in \mathbb{N}$. If $A \in \mathcal{B}(\mathcal{H})$ is both hyponormal and accretive, then A is also 2-hyperaccretive, as $A^{*2} + 2A^*A + A^2 \geqslant A^{*2} + A^*A + AA^* + A^2 = (A^* + A)^2 \geqslant 0$.

Next we complement [9, Lemmas 2.5 and 2.7] with some basic properties of hyperaccretive operators.

Theorem 2.4: *If $A \in \mathcal{B}(\mathcal{H})$ is accretive, $J \in \mathbb{N}_\circ$, $K, M, N \in \mathbb{N}$, $K \leqslant N < M$ and $T > 0$, then:*

(a) *there exists* $\mho^-_{A^*,A}(I) \stackrel{\text{def}}{=} {}^s\!\lim_{t\to\infty} \mathrm{e}^{-tA^*}\,\mathrm{e}^{-tA} = A_{\mathrm{e}^{-A}} = A_{\mathrm{e}^{-TA}}$,
(b) $\int_{\mathbb{R}_+} \mathrm{e}^{-tA^*}(A^*+A)\,\mathrm{e}^{-tA}\,\mathrm{d}t = I - \mho^-_{A^*,A}(I)$,
(c) $(A^*\otimes I + I\otimes A)\mho^-_{A^*,A}(I) = A^*\mho^-_{A^*,A}(I) + \mho^-_{A^*,A}(I)A = 0$, *i.e.* $\mho^-_{A^*,A}(I) \in \mathcal{N}(A^*\otimes I + I\otimes A)$ *if A is 2-hyperaccretive,*
(d) *if A is also N-accretive for $N \geqslant 2$, then*

$$\sum_{n=0}^{N-1}\binom{N-1}{n}A^{*n}A^{N-1-n} = \int_{\mathbb{R}_+} \mathrm{e}^{-tA^*}\left(\sum_{n=0}^{N}\binom{N}{n}A^{*n}A^{N-n}\right)\mathrm{e}^{-tA}\,\mathrm{d}t \geqslant 0, \qquad (6)$$

(e) *an accretive operator in* $\mathfrak{B}(\mathcal{H})$ *is* N*-hyperaccretive if and only if it is* N*-accretive,*

(f) $\left\{\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{N-1}(I)\,\mathrm{e}^{-tA}\right\}_{t\geqslant 0}$ *monotonically decreases, strongly converges to* $\mho_{A^*,A}^{-}\left(\Delta_{A^*,A}^{N-1}(I)\right)$ $=\Delta_{A^*,A}^{N-1}\left(\mho_{A^*,A}^{-}(I)\right)$ *as* $t\to+\infty$ *and* $\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{N-1}(I)\,\mathrm{e}^{-tA}\geqslant\Delta_{A^*,A}^{N-1}\left(\mho_{A^*,A}^{-}(I)\right)$, *if* A *is* N*-accretive,*

(g) *moreover, if* A *is* N*-accretive for some* $N\geqslant 2$, *it follows*

$$\frac{1}{(N-1)!}\int_{\mathbb{R}_+} t^{N-1}\,\mathrm{e}^{-tA^*}\left(\sum_{n=0}^{N}\binom{N}{n}A^{*n}A^{N-n}\right)\mathrm{e}^{-tA}\,\mathrm{d}t = I-\mho_{A^*,A}^{-}(I), \tag{7}$$

$$\frac{1}{(N-1)!}\int_{\mathbb{R}_+} t^{N-1}\,\mathrm{e}^{-tA^*}\left(\sum_{m=0}^{M}\binom{M}{m}A^{*m}A^{M-m}\right)\mathrm{e}^{-tA}\,\mathrm{d}t = \sum_{k=0}^{M-N}\binom{M-N}{k}A^{*k}A^{M-N-k}, \tag{8}$$

$$\begin{aligned}\operatorname*{{}^s\!\lim}_{T\to+\infty}\sqrt{\frac{T^k}{k!}\Delta_{A^*,A}^{k+J}(I)}\,\mathrm{e}^{-TA} &= \operatorname*{{}^s\!\lim}_{T\to+\infty}\frac{T^k}{k!}\,\mathrm{e}^{-TA^*}\Delta_{A^*,A}^{k+J}(I)\,\mathrm{e}^{-TA}\\ &= \operatorname*{{}^w\!\lim}_{T\to+\infty}\,\mathrm{e}^{-TA^*}\sqrt{\frac{T^k}{k!}\Delta_{A^*,A}^{k+J}(I)}=0,\end{aligned} \tag{9}$$

for all $\mathbb{N}\ni k\leqslant N-1$. *If* A *is also normal, then* ${}^w\!\lim_{T\to+\infty}$ *in (9) can be replaced by* ${}^s\!\lim_{T\to+\infty}$.

Proof: The first equality in (a) follows by Jocić et al. [9, Lemma 2.5(e)], while the remaining statements in (a) derives directly.

The application of the formula [9, (9)] to (A^*,A,I,t) instead of (A,B,X,T) gives $\int_0^t \mathrm{e}^{-sA^*}(A^*+A)\,\mathrm{e}^{-sA}ds = I-\,\mathrm{e}^{-tA^*}\mathrm{e}^{-tA}$, so by letting $t\to+\infty$ the formula (b) follows.

The proof for (c) follows from $(A^*\otimes I+I\otimes A)\mho_{A^*,A}^{-}(I)={}^s\!\lim_{t\to+\infty}\,\mathrm{e}^{-tA^*}(A^*+A)\,\mathrm{e}^{-tA}$ $=0$, according to [9, Lemma 2.5(e)].

The proof for (d) relies on the following equalities:

$$\begin{aligned}\Delta_{A^*,A}^{N-1}(I) &= \Delta_{A^*,A}^{N-1}(I-\mho_{A^*,A}^{-}(I)) = \Delta_{A^*,A}^{N-1}\left(\int_{\mathbb{R}_+}\mathrm{e}^{-tA^*}(A^*+A)\,\mathrm{e}^{-tA}\,\mathrm{d}t\right)\\ &= \int_{\mathbb{R}_+}\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{N-1}(A^*+A)\,\mathrm{e}^{-tA}\,\mathrm{d}t = \int_{\mathbb{R}_+}\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{N}(I)\,\mathrm{e}^{-tA}\,\mathrm{d}t.\end{aligned} \tag{10}$$

The first equality in (10) is based on the property (c), while the second one in (10) is due to the formula (b).

Property (e) follows by the iterated applications of (d).

To prove (f), let us note that for $0\leqslant s\leqslant t$ it follows from the formula (2), applied to $(A^*,A,\Delta_{A^*,A}^{N-1}(I))$ instead of (A,B,X),

$$\mathrm{e}^{-sA^*}\Delta_{A^*,A}^{N-1}(I)\,\mathrm{e}^{-sA}-\,\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{N-1}(I)\,\mathrm{e}^{-tA}=\int_{[s,t]}\mathrm{e}^{-uA^*}\Delta_{A^*,A}^{N}(I)\,\mathrm{e}^{-uA}\,\mathrm{d}u\geqslant 0$$

as $\Delta^N_{A^*,A}(I) \geqslant 0$, which proves the monotonicity of the considered operator family and the existence of ${}^s\lim_{t\to+\infty} \mathrm{e}^{-tA^*}\Delta^{N-1}_{A^*,A}(I)\,\mathrm{e}^{-tA}$. Moreover, for all $t \geqslant 0$

$$\begin{aligned}
\mathrm{e}^{-tA^*}\Delta^{N-1}_{A^*,A}(I)\,\mathrm{e}^{-tA} &\geqslant \mho^-_{A^*,A}\left(\Delta^{N-1}_{A^*,A}(I)\right) = \underset{t\to+\infty}{{}^s\lim}\ \mathrm{e}^{-tA^*}\Delta^{N-1}_{A^*,A}(I)\,\mathrm{e}^{-tA}\\
&= \underset{t\to+\infty}{{}^s\lim}\ \mathrm{e}^{-t\Delta_{A^*,A}}\Delta^{N-1}_{A^*,A}(I) = \underset{t\to+\infty}{{}^s\lim}\ \Delta^{N-1}_{A^*,A}\,\mathrm{e}^{-t\Delta_{A^*,A}}(I) = \underset{t\to+\infty}{{}^s\lim}\ \Delta^{N-1}_{A^*,A}\left(\mathrm{e}^{-tA^*}\mathrm{e}^{-tA}\right)\\
&= \Delta^{N-1}_{A^*,A}\left(\underset{t\to+\infty}{{}^s\lim}\ \mathrm{e}^{-tA^*}\mathrm{e}^{-tA}\right) = \Delta^{N-1}_{A^*,A}\left(\mho^-_{A^*,A}(I)\right),
\end{aligned}$$

as $\Delta_{A^*,A}$ and $\Delta^{N-1}_{A^*,A}$ are transformers which are continuous in the strong operator topology.

The proof for the formula (7) in (g) will be based on induction, so starting from the inductive hypothesis that the formula (7) holds for $N-1$ instead of N, for an arbitrary $h \in \mathcal{H}$ it follows

$$\begin{aligned}
\langle (I-\mho^-_{A^*,A}(I))h,h\rangle &= \frac{1}{(N-2)!}\int_{\mathbb{R}_+} t^{N-2}\left\langle \mathrm{e}^{-tA^*}\Delta^{N-1}_{A^*,A}(I)\,\mathrm{e}^{-tA}h,h\right\rangle \mathrm{d}t\\
&= \frac{1}{(N-2)!}\int_{\mathbb{R}_+} t^{N-2}\left\langle \mathrm{e}^{-tA^*}\int_{\mathbb{R}_+} \mathrm{e}^{-sA^*}\Delta^{N}_{A^*,A}(I)\,\mathrm{e}^{-sA}\,\mathrm{d}s\,\mathrm{e}^{-tA}h,h\right\rangle \mathrm{d}t \qquad (11)\\
&= \frac{1}{(N-2)!}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} t^{N-2}\left\langle \mathrm{e}^{-(t+s)A^*}\Delta^{N}_{A^*,A}(I)\,\mathrm{e}^{-(t+s)A}h,h\right\rangle \mathrm{d}t\,\mathrm{d}s\\
&= \frac{1}{(N-2)!}\int_{\mathbb{R}_+}\int_0^u t^{N-2}\,\mathrm{d}t\left\langle \mathrm{e}^{-uA^*}\Delta^{N}_{A^*,A}(I)\,\mathrm{e}^{-uA}h,h\right\rangle \mathrm{d}u\\
&= \left\langle\left(\frac{1}{(N-1)!}\int_{\mathbb{R}_+} u^{N-1}\,\mathrm{e}^{-uA^*}\Delta^{N}_{A^*,A}(I)\,\mathrm{e}^{-uA}\mathrm{d}u\right)h,h\right\rangle, \qquad (12)
\end{aligned}$$

which actually proves the formula (7). The last equality in (11) is due to the formula (6), while the last equality in (12) follows by the change of variables $u := s+t$, combined with the double application of the Fubini theorem.

The proof for the formula (8) follows from

$$\begin{aligned}
&\frac{1}{(N-1)!}\int_{\mathbb{R}_+} t^{N-1}\,\mathrm{e}^{-tA^*}\,\Delta^{M}_{A^*,A}(I)\,\mathrm{e}^{-tA}\,\mathrm{d}t\\
&\quad= \frac{1}{(N-1)!}\Delta^{M-N}_{A^*,A}\left(\int_{\mathbb{R}_+} t^{N-1}\,\mathrm{e}^{-tA^*}\,\Delta^{N}_{A^*,A}(I)\,\mathrm{e}^{-tA}\,\mathrm{d}t\right) = \Delta^{M-N}_{A^*,A}(I-\mho^-_{A^*,A}(I))\\
&\quad= \Delta^{M-N}_{A^*,A}(I), \qquad (13)
\end{aligned}$$

where the second equality in (13) is based on the formula (7), while the remaining equality in (13) is due to the property presented in (c).

To prove that the second expression in the formula (9) equals to zero, for an arbitrary $\mathbb{N} \ni k \leqslant N-1$, we rely on the following inequality-equality chain

$$\underset{T\to+\infty}{{}^s\lim}\ \frac{T^k}{k!}\,\mathrm{e}^{-TA^*}\,\Delta^{k+J}_{A^*,A}(I)\,\mathrm{e}^{-TA} \leqslant \underset{T\to+\infty}{{}^s\lim}\ \sum_{k=1}^{N-1}\frac{T^k}{k!}\,\mathrm{e}^{-TA^*}\,\Delta^{k+J}_{A^*,A}(I)\,\mathrm{e}^{-TA}$$

$$= \Delta^{J}_{A^*\!,A}\left({}^{s}\!\!\lim_{T\to+\infty}\left(I - \mathrm{e}^{-TA^*}\mathrm{e}^{-TA} - \frac{1}{(N-1)!}\int_{[0,T]} t^{N-1}\,\mathrm{e}^{-tA^*}\Delta^{N}_{A^*\!,A}(I)\,\mathrm{e}^{-tA}\,\mathrm{d}t\right)\right) \tag{14}$$

$$= \Delta^{J}_{A^*\!,A}\left(I - \mho^{-}_{A^*\!,A}(I) - \left(I - \mho^{-}_{A^*\!,A}(I)\right)\right) = 0. \tag{15}$$

The equality in (14) is based on the formula (6) in [9, Lemma 2.4] for $n := N$, while the first equality in (15) is due to the formula (7). Therefore, the first equality in (9) follows from the fact that $\lim_{T\to+\infty}\left\|\sqrt{\frac{T^k}{k!}\Delta^{k+J}_{A^*\!,A}(I)}\,\mathrm{e}^{-TA}h\right\|^2 = \lim_{T\to+\infty}\left\langle \frac{T^k}{k!}\,\mathrm{e}^{-TA^*}\Delta^{k+J}_{A^*\!,A}(I)\,\mathrm{e}^{-TA}h, h\right\rangle =$ 0 for all $h \in \mathcal{H}$. The last equality in (9) follows from the previous one simply because $\lim_{T\to+\infty}\left\langle \mathrm{e}^{-TA^*}\sqrt{\frac{T^k}{k!}\Delta^{k+J}_{A^*\!,A}(I)}g, h\right\rangle = \lim_{T\to+\infty}\left\langle g, \sqrt{\frac{T^k}{k!}\Delta^{k+J}_{A^*\!,A}(I)}\,\mathrm{e}^{-TA}h\right\rangle = 0$ for all $g, h \in$ $\mathcal{H}$. The last remark in (g) is based on the equality $\lim_{T\to+\infty}\left\|\mathrm{e}^{-TA^*}\left(\frac{T^k}{k!}\Delta^{k+J}_{A^*\!,A}(I)\right)^{\frac{1}{2}} h\right\|^2 =$ $\lim_{T\to+\infty}\left\|\left(\frac{T^k}{k!}\Delta^{k+J}_{A^*\!,A}(I)\right)^{\frac{1}{2}}\,\mathrm{e}^{-TA}h\right\|^2 = 0$ for all $h \in \mathcal{H}$, whenever A is normal. ■

In the sequel, we will use the following extended standard multi-index notations.

Definition 2.5: For an arbitrary $N \in \mathbb{N}$ we define $\boldsymbol{k} \stackrel{\text{def}}{=} (k_1, \ldots, k_N) \in \{0,1\}^N$, $\boldsymbol{c} \stackrel{\text{def}}{=} (c_1, \ldots, c_N) \in [0,+\infty)^N$ and $\boldsymbol{c}^{\boldsymbol{k}} \stackrel{\text{def}}{=} \prod_{n=1}^{N} c_n^{k_n}$, together with the convention $0^0 \stackrel{\text{def}}{=} 1$ and $|\boldsymbol{k}| \stackrel{\text{def}}{=} \sum_{n=1}^{N} k_n$.

Lemma 2.6: *If $N \in \mathbb{N}$, $A, B^* \in \mathcal{B}(\mathcal{H})$ are N-hyperaccretive, $\{c_n\}_{n=1}^{N}$ is a sequence in $[0,+\infty)$, $c_\circ > 0$ and $P_N\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto c_\circ \prod_{n=1}^{N}(c_n + z)$, then*

$$P_N(A^* \otimes I + I \otimes A)(I) \geqslant 0, \quad P_N(B \otimes I + I \otimes B^*)(I) \geqslant 0. \tag{16}$$

Consequently, A is N-hyperaccretive if and only if $A + \varepsilon I$ is N-hyperaccretive for all $\varepsilon > 0$.

Proof: As $\sum_{k=0}^{N-n}\binom{N-n}{k}A^{*k}A^{N-n-k} \geqslant 0$ for all $0 \leqslant n \leqslant N$, then the proof is based on the following chain of equalities and inequalities:

$$P_N(A^* \otimes I + I \otimes A)(I) = c_\circ \prod_{n=1}^{N}(c_n I \otimes I + A^* \otimes I + I \otimes A)(I)$$

$$= c_\circ \sum_{n=0}^{N}\sum_{|\boldsymbol{k}|=n}\boldsymbol{c}^{\boldsymbol{k}}(A^* \otimes I + I \otimes A)^{N-n}(I) = c_\circ \sum_{n=0}^{N}\sum_{|\boldsymbol{k}|=n}\boldsymbol{c}^{\boldsymbol{k}}\sum_{k=0}^{N-n}\tbinom{N-n}{k}A^{*k}A^{N-n-k},$$

so $P_N(A^* \otimes I + I \otimes A)(I) \geqslant 0$ as $d_n := c_\circ \sum_{|\boldsymbol{k}|=n}\boldsymbol{c}^{\boldsymbol{k}} \geqslant 0$ for all $\mathbb{N} \ni n \leqslant N$. The remaining inequality in (16) follows from those already obtained, simply by taking $A := B^*$.

The special case $c_\circ := 1$ and $c_k := \varepsilon$ for all $\mathbb{N} \ni k \leqslant K \leqslant N$ of the inequality (16) shows that $A + \varepsilon I$ is N-hyperaccretive for all $\varepsilon > 0$ if A is N-hyperaccretive (see also [9, Lemma 2.2]). Conversely, $(A^* \otimes I + I \otimes A)^K(I) = {}^{\mathcal{B}(\mathcal{H})}\!\lim_{\varepsilon\searrow 0}\left((A+\varepsilon I)^* \otimes I + I \otimes (A+\varepsilon I)\right)^K(I) \geqslant 0$ if $\mathbb{N} \ni K \leqslant N$ and $A + \varepsilon I$ is N-hyperaccretive for all $\varepsilon > 0$. ■

Note that in the special case $c_\circ := 1$ and $c_n := 0$ for all $1 \leqslant n \leqslant N$ the inequality (16) reduces to the very definition of N-hyperaccretivity of operators.

3. Main results

3.1. $\int_0^{+\infty} \sqrt{A^* + A}\, \mathrm{e}^{-t(A^*\otimes I + I\otimes B)} \sqrt{B + B^*}\, \mathrm{d}t$ transformers

The impact of accretivity to the standard normal extension of subnormal operator and its consequences is the subject of the following lemma.

Lemma 3.1: *Let Φ, Ψ be a s.n. functions, $p \geqslant 2$ and $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that A, B are accretive and $X \in \mathcal{C}_\Psi(\mathcal{H})$.*

(a) *If, in addition, A is quasinormal operator with injective real part, such that A^* is 2-hyperaccretive, then*
   (a) (a1)$\tilde{A} := \begin{bmatrix} A & \sqrt{A^*A - AA^*} \\ 0 & A^* \end{bmatrix}$ *is (the standard) normal accretive extension of A to $\mathcal{H} \oplus \mathcal{H}$ and the family $\left\{\sqrt{\tilde{A} + \tilde{A}^*}\, \mathrm{e}^{-t\tilde{A}}\right\}_{t \geqslant 0}$ is a m.c.n.o. and s.i. family on $\mathbb{R}_+$ (in respect to the Lebesgue measure),*
   (b) (a2)*the following formulas hold:*

$$\begin{aligned}
&\sqrt{\tilde{A}^* + \tilde{A}} \\
&= \begin{bmatrix} A^* + A & \sqrt{A^*A - AA^*} \\ \sqrt{A^*A - AA^*} & A^* + A \end{bmatrix}^{\frac{1}{2}} \\
&= \left\| \begin{bmatrix} \sqrt{A^* + A} & (A^* + A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*} \\ 0 & \sqrt{I - \left((A^* + A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*} (A^* + A)^{-\frac{1}{2}}\right)^2} \sqrt{A^* + A} \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} \sqrt{I - \left((A^* + A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*} (A^* + A)^{-\frac{1}{2}}\right)^2} \sqrt{A^* + A} & 0 \\ (A^* + A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*} & \sqrt{A^* + A} \end{bmatrix} \right\|.
\end{aligned} \tag{17}$$

(b)

$$\left\| \int_{\mathbb{R}_+} \sqrt{A^* + A} \mathrm{e}^{-tA} X \,\mathrm{e}^{-tB} \sqrt{B + B^*}\, \mathrm{d}t \right\|_\Psi \leqslant \|X\|_\Psi, \tag{18}$$

hold under any of the following conditions:
   (a) (b1)$\Psi := \ell^1$,
   (b) (b2)*if $\Psi := \Phi^{(p)*}$ and A or B^* is quasinormal operator having the adjoint operator which is 2-hyperaccretive operator with injective real part,*
   (c) (b3)*if both A and B^* are quasinormal operators having its adjoint operators which are 2-hyperaccretive operators with injective real parts.*
   (d) (c1)*If $\Psi := \Phi^{(p)}$, A is quasinormal operator having the adjoint operator which is 2-hyperaccretive operator with injective real part and B is strictly accretive, then*

$$\left\| \int_{\mathbb{R}_+} \sqrt{A^* + A}\, \mathrm{e}^{-tA} X \,\mathrm{e}^{-tB}\, \mathrm{d}t \right\|_{\Phi^{(p)}} \leqslant \|X(B + B^*)^{-1/2}\|_{\Phi^{(p)}}. \tag{19}$$

(e) (c2)*If* $\Psi := \Phi^{(p)}$, *A is strictly accretive and* B^* *is quasinormal operator having the adjoint operator which is 2-hyperaccretive operator with injective real part, then*

$$\left\| \int_{\mathbb{R}_+} \mathrm{e}^{-tA} X \, \mathrm{e}^{-tB} \sqrt{B+B^*} \, \mathrm{d}t \right\|_{\Phi^{(p)}} \leqslant \|(A^*+A)^{-1/2} X\|_{\Phi^{(p)}}. \tag{20}$$

Proof: To prove (a1), let us note that the requirement $A^2 + 2AA^* + A^{*2} \geqslant 0$ is equivalent to $\left(\sqrt{A^*A - AA^*}\right)^2 = A^*A - AA^* \leqslant (A^*+A)^2$, which by Heinz inequality implies $\sqrt{A^*A - AA^*} \leqslant A^*+A$, and $0 \leqslant C_A := (A^*+A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*}(A^*+A)^{-\frac{1}{2}} \leqslant I$, if A is strictly accretive. In the general case, if A^*+A is injective, then we define a contractive operator $\widetilde{C}_A : \mathcal{R}(\sqrt{A^*+A}) \to \mathcal{H}: \sqrt{A^*+A} f \mapsto (A^*+A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*} f$ for every $f \in \mathcal{H}$, which can be uniquely extended to the contraction on the whole $\overline{\mathcal{R}(\sqrt{A^*+A})} = \overline{\mathcal{R}(A^*+A)} = \mathcal{H}$ and also denoted by C_A or $(A^*+A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*}(A^*+A)^{-\frac{1}{2}}$. In the same manner, by $(A^*+A)^{-\frac{1}{2}} \sqrt{A^*A - AA^*}$ will be denoted a bounded operator $C_A \sqrt{A^*+A}$. Thus C_A is a contraction in both cases and therefore $\begin{bmatrix} I & C_A \\ C_A & I \end{bmatrix} \geqslant \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ according to [10, Lemma 3.1].

The first two equalities in (17) of (a2) are based on the chain of identities

$$\begin{aligned}
0 &\leqslant \left\| \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I - C_A^2}\sqrt{A^*+A} \end{bmatrix} \right\|^2 \\
&= \begin{bmatrix} \sqrt{A^*+A} & 0 \\ \sqrt{A^*+A}C_A & \sqrt{A^*+A}\sqrt{I - C_A^2} \end{bmatrix} \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I - C_A^2}\sqrt{A^*+A} \end{bmatrix} \\
&= \begin{bmatrix} A^*+A & \sqrt{A^*+A}C_A\sqrt{A^*+A} \\ \sqrt{A^*+A}C_A\sqrt{A^*+A} & \sqrt{A^*+A}C_A^2\sqrt{A^*+A} + \sqrt{A^*+A}(I - C_A^2)\sqrt{A^*+A} \end{bmatrix} \\
&= \begin{bmatrix} A^*+A & \sqrt{A^*A - AA^*} \\ \sqrt{A^*A - AA^*} & A^*+A \end{bmatrix} = \tilde{A}^* + \tilde{A},
\end{aligned}$$

which also proves that $\tilde{A}$ is an accretive normal operator and $\breve{A} \stackrel{\text{def}}{=} \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I - C_A^2}\sqrt{A^*+A} \end{bmatrix}$ satisfies $\sqrt{\tilde{A}^* + \tilde{A}} = |\breve{A}|$. Therefore, $\left\{ \sqrt{\tilde{A}^* + \tilde{A}} \, \mathrm{e}^{-t\tilde{A}} \right\}_{t \geqslant 0}$ is a m.c.n.o. family, which according to Theorem 2.4(b) is also a s.i. family, completing the proof for (a1).

Similarly, to prove the third equality in (17) let $\hat{A} \stackrel{\text{def}}{=} \begin{bmatrix} \sqrt{I - C_A^2}\sqrt{A^*+A} & 0 \\ C_A\sqrt{A^*+A} & \sqrt{A^*+A} \end{bmatrix}$, so

$$\begin{aligned}
|\hat{A}|^2 &= \left\| \begin{bmatrix} \sqrt{I - C_A^2}\sqrt{A^*+A} & 0 \\ C_A\sqrt{A^*+A} & \sqrt{A^*+A} \end{bmatrix} \right\|^2 \\
&= \begin{bmatrix} \sqrt{A^*+A}\sqrt{I - C_A^2} & \sqrt{A^*+A}C_A \\ 0 & \sqrt{A^*+A} \end{bmatrix} \begin{bmatrix} \sqrt{I - C_A^2}\sqrt{A^*+A} & 0 \\ C_A\sqrt{A^*+A} & \sqrt{A^*+A} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{A^*+A}(I - C_A^2 + C_A^2)\sqrt{A^*+A} & \sqrt{A^*+A}C_A\sqrt{A^*+A} \\ \sqrt{A^*+A}C_A\sqrt{A^*+A} & A^*+A \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} A^*+A & \sqrt{A^*A-AA^*} \\ \sqrt{A^*A-AA^*} & A^*+A \end{bmatrix} = \tilde{A}^* + \tilde{A}.$$

To prove (b1), we apply the inequality (30) in [5, Th. 3.1(c)] to s.i. families $\{A_t\}_{t\in\mathbb{R}_+}$ and $\{B_t^*\}_{t\in\mathbb{R}_+}$ defined by $A_t \overset{\text{def}}{=} \sqrt{A^*+A}\,\mathrm{e}^{-tA}$ and $B_t \overset{\text{def}}{=} \mathrm{e}^{-tB}\sqrt{B+B^*}$ for all $t \geqslant 0$, based on the facts that $\int_{\mathbb{R}_+} \mathrm{e}^{-tA^*}(A^*+A)\,\mathrm{e}^{-tA}\,\mathrm{d}t \leqslant I$ and $\int_{\mathbb{R}_+} \mathrm{e}^{-tB}(B+B^*)\,\mathrm{e}^{-tB^*}\mathrm{d}t \leqslant I$, according to the Theorem 2.4(b) and the inequality (1).

To prove (b3), note that $\widetilde{B^*} = \left[\begin{smallmatrix} B^* & \sqrt{BB^*-B^*B} \\ 0 & B \end{smallmatrix}\right]$ and let $\check{X} := \left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$, $\{V_t\}_{t\in\mathbb{R}_+}$ and $\{W_t\}_{t\in\mathbb{R}_+}$ be operators $V_t \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \sum_{k=0}^{n-1} A^{n-k-1}\sqrt{A^*A-AA^*}A^{*k}$ and $W_t \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \sum_{k=0}^{n-1} B^{*k}\sqrt{BB^*-B^*B}B^{n-k-1}$ for all $t \geqslant 0$, which enables representations

$$\mathrm{e}^{-t\tilde{A}} = \begin{bmatrix} \mathrm{e}^{-tA} & V_t \\ 0 & \mathrm{e}^{-tA^*} \end{bmatrix} \quad \text{and} \quad \mathrm{e}^{-t\widetilde{B^*}} = \begin{bmatrix} \mathrm{e}^{-tB^*} & W_t \\ 0 & \mathrm{e}^{-tB} \end{bmatrix}.$$

Thus we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}_+} \sqrt{A^*+A}\,\mathrm{e}^{-tA} X\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t \right\|_\Psi \\
&= \left\| \begin{bmatrix} 0 & \int_{\mathbb{R}_+}\sqrt{A^*+A}\,\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t \\ 0 & 0 \end{bmatrix} \right\|_\Psi && (21)\\
&= \left\| \int_{\mathbb{R}_+} \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I-C_A^2}\sqrt{A^*+A} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-tA} & V_t \\ 0 & \mathrm{e}^{-tA^*} \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \times \left. \begin{bmatrix} \mathrm{e}^{-tB^*} & W_t \\ 0 & \mathrm{e}^{-tB} \end{bmatrix} \begin{bmatrix} \sqrt{B+B^*}\sqrt{I-C_{B^*}^2} & \sqrt{B+B^*}C_{B^*} \\ 0 & \sqrt{B+B^*} \end{bmatrix} \mathrm{d}t \right\|_\Psi \\
&= \left\| U_{\check{A}} \int_{\mathbb{R}_+} \left| \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I-C_A^2}\sqrt{A^*+A} \end{bmatrix} \right| \mathrm{e}^{-t\tilde{A}}\check{X}\mathrm{e}^{-t\widetilde{B^*}} \right. \\
&\quad \left. \left| \begin{bmatrix} \sqrt{B+B^*}\sqrt{I-C_{B^*}^2} & \sqrt{B+B^*}C_{B^*} \\ 0 & \sqrt{B+B^*} \end{bmatrix}^* \right| \mathrm{d}t\, U_{\widehat{B^*}}{}^* \right\|_\Psi && (22)\\
&\leqslant \left\| \int_{\mathbb{R}_+} \sqrt{\tilde{A}^*+\tilde{A}}\,\mathrm{e}^{-t\tilde{A}}\check{X}\,\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*}+\widetilde{B^*}^*}\,\mathrm{d}t \right\|_\Psi && (23)\\
&\leqslant \left\| \left(\int_{\mathbb{R}_+} \mathrm{e}^{-t\tilde{A}^*}\left(\tilde{A}^*+\tilde{A}\right)\mathrm{e}^{-t\tilde{A}}\,\mathrm{d}t \right)^{\frac{1}{2}} \check{X} \left(\int_{\mathbb{R}_+} \mathrm{e}^{-t\widetilde{B^*}}\left(\widetilde{B^*}+\widetilde{B^*}^*\right)\mathrm{e}^{-t\widetilde{B^*}^*}\mathrm{d}t \right)^{\frac{1}{2}} \right\|_\Psi \\
&\leqslant \|\check{X}\|_\Psi = \|X\|_\Psi. && (24)
\end{aligned}$$

The equality in (21) is a consequence of the equality for singular values of the operator and operator matrix involved therein, while the last equality in (22) follows from

$$\begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I-C_A^2}\sqrt{A^*+A} \end{bmatrix} = \check{A}$$

$$= U_{\breve{A}} \left\| \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I-C_A^2}\sqrt{A^*+A} \end{bmatrix} \right\| = U_{\breve{A}} \sqrt{\widetilde{A}^* + \widetilde{A}}, \tag{25}$$

$$\begin{bmatrix} \sqrt{I-C_{B^*}^2}\sqrt{B+B^*} & 0 \\ C_{B^*}\sqrt{B+B^*} & \sqrt{B+B^*} \end{bmatrix} = \widehat{B^*} = U_{\widehat{B^*}} \left\| \begin{bmatrix} \sqrt{I-C_{B^*}^2}\sqrt{B+B^*} & 0 \\ C_{B^*}\sqrt{B+B^*} & \sqrt{B+B^*} \end{bmatrix} \right\|$$
$$= U_{\widehat{B^*}} \sqrt{\widetilde{B^*}^* + \widetilde{B^*}},$$
$$\begin{bmatrix} \sqrt{B+B^*}\sqrt{I-C_{B^*}^2} & \sqrt{B+B^*}C_{B^*} \\ 0 & \sqrt{B+B^*} \end{bmatrix} = \begin{bmatrix} \sqrt{I-C_{B^*}^2}\sqrt{B+B^*} & 0 \\ C_{B^*}\sqrt{B+B^*} & \sqrt{B+B^*} \end{bmatrix}^*$$
$$= \sqrt{\widetilde{B^*}^* + \widetilde{B^*}}\, U_{\widehat{B^*}}{}^*, \tag{26}$$

which provides formulas for the canonical polar representations for $\breve{A}$ and $\widehat{B^*}$, where $U_{\breve{A}}$ and $U_{\widehat{B^*}}$ are canonical partial isometries on $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$.

The inequality in (23) follows by a double application of formulas in (17), the first inequality in (24) is based on the Cauchy–Schwarz inequality (23) in [3, Th. 3.2], while the second inequality in (24) follows by the application of Theorem 2.4(b) property, combined with the inequality (1).

Finally, the proof of (b2) relies on the inequality (32) in [5, Th. 3.1(d)], by modifying the proof for the case (b3). So, if A is quasinormal and A^* is 2-hyperaccretive with injective real part, we can take $\tilde{A} := \left[\begin{smallmatrix} A & \sqrt{A^*A-AA^*} \\ 0 & A^* \end{smallmatrix}\right]$ and $0 \oplus B := \left[\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right]$ to apply the inequality (32) in [5, Th. 3.1(d)] to s.i. and m.c.n.o. family $\boldsymbol{A} := \left\{ \left(\widetilde{A}^* + \widetilde{A}\right)^{1/2} \mathrm{e}^{-t\tilde{A}} \right\}_{t\geqslant 0}$ and s.i. family $\boldsymbol{B}^* := \left\{ 0 \oplus \sqrt{B+B^*}\,\mathrm{e}^{-tB^*} \right\}_{t\geqslant 0}$. The complementing case in which B^* is quasinormal and B is 2-hyperaccretive with injective real part proves similarly.

To prove the inequality (19) in (c1), let us take again $\tilde{A} := \left[\begin{smallmatrix} A & \sqrt{A^*A-AA^*} \\ 0 & A^* \end{smallmatrix}\right]$ and $0 \oplus B := \left[\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right]$ to get s.i. and m.c.n.o. family $\boldsymbol{A} := \left\{ \left(\widetilde{A}^* + \widetilde{A}\right)^{1/2} \mathrm{e}^{-t\tilde{A}} \right\}_{t\geqslant 0}$ and s.i. family $\boldsymbol{B} := \left\{ 0 \oplus \sqrt{B^*+B}\,\mathrm{e}^{-tB} \right\}_{t\geqslant 0}$. By denoting $Y := X(B+B^*)^{-1/2}$ we get

$$\left\| \int_{\mathbb{R}_+} \sqrt{A^*+A}\,\mathrm{e}^{-tA} X \mathrm{e}^{-tB}\,\mathrm{d}t \right\|_{\Phi^{(p)}} = \left\| \int_{\mathbb{R}_+} \sqrt{A^*+A}\,\mathrm{e}^{-tA} Y \sqrt{B^*+B}\,\mathrm{e}^{-tB}\,\mathrm{d}t \right\|_{\Phi^{(p)}}$$
$$= \left\| \begin{bmatrix} 0 & \int_{\mathbb{R}_+} \sqrt{A^*+A}\,\mathrm{e}^{-tA} Y \sqrt{B+B^*}\,\mathrm{e}^{-tB}\,\mathrm{d}t \\ 0 & 0 \end{bmatrix} \right\|_{\Phi^{(p)}}$$
$$= \left\| \int_{\mathbb{R}_+} \begin{bmatrix} \sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I-C_A^2}\sqrt{A^*+A} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-tA} & V_t \\ 0 & \mathrm{e}^{-tA^*} \end{bmatrix} \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \right.$$
$$\left. \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{B^*+B} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{e}^{-tB} \end{bmatrix} \mathrm{d}t \right\|_{\Phi^{(p)}}$$

$$= \left\| U_{\check{A}} \int_{\mathbb{R}_+} \sqrt{\tilde{A}^* + A}\, \mathrm{e}^{-t\tilde{A}} \check{Y} \left(0 \oplus \sqrt{B^* + B}\, \mathrm{e}^{-tB}\right) \mathrm{d}t \right\|_{\Phi^{(p)}} \tag{27}$$

$$\leqslant \|U_{\check{A}}\| \left\| \left(\int_{\mathbb{R}_+} \mathrm{e}^{-t\tilde{A}^*} \left(\tilde{A}^* + \tilde{A}\right) \mathrm{e}^{-t\tilde{A}} \right)^{1/2} \check{Y} \right\|_{\Phi^{(p)}} \left\| 0 \oplus \int_{\mathbb{R}_+} \mathrm{e}^{-tB^*} (B^* + B)\, \mathrm{e}^{-tB}\, \mathrm{d}t \right\|^{1/2} \tag{28}$$

$$\leqslant \left\| I^{1/2} \check{Y} \right\|_{\Phi^{(p)}} \|0 \oplus I\|^{1/2} = \|Y\|_{\Phi^{(p)}} = \|X(B + B^*)^{-1/2}\|_{\Phi^{(p)}}, \tag{29}$$

where the (last) equality in (27) is based on the polar decomposition formula (25), while the inequality (28) is obtained by using the Cauchy–Schwarz inequality (28) in [5, Th. 3.1(a)] to families $\boldsymbol{A}$, $\boldsymbol{B}$ and to Y instead of X therein. The inequality in (29) is due to [9, Lemma 2.5(c)], combined with the double monotonicity property (1).

The just proven part (c1) enables to prove the inequality (20) in (c2) as

$$\left\| \int_{\mathbb{R}_+} \mathrm{e}^{-tA} X\, \mathrm{e}^{-tB} \sqrt{B + B^*}\, \mathrm{d}t \right\|_{\Phi^{(p)}} = \left\| \left(\int_{\mathbb{R}_+} \mathrm{e}^{-tA} X\, \mathrm{e}^{-tB} \sqrt{B + B^*}\, \mathrm{d}t \right)^* \right\|_{\Phi^{(p)}} \tag{30}$$

$$= \left\| \int_{\mathbb{R}_+} \sqrt{B + B^*}\, \mathrm{e}^{-tB^*} X^* \mathrm{e}^{-tA^*}\, \mathrm{d}t \right\|_{\Phi^{(p)}} \leqslant \|X^*(A^* + A)^{-1/2}\|_{\Phi^{(p)}}$$

$$= \|(A^* + A)^{-1/2} X\|_{\Phi^{(p)}}, \tag{31}$$

where the equality in (30) and the last equality in (31) are based on the adjoint property for u.i. norm $\|\cdot\|_{\Phi^{(p)}}$, while the inequality in (31) follows by the inequality (19) applied to (B^*, A^*, X^*) instead of (A, B, X). ■

Moreover, the next theorem complements [9, Lemma 2.7] and provides the extension of the arithmetic-geometric u.i. norm inequality for normal accretive operators in [9, Th. 2.9(b1),(b2),(d),(e)] to quasinormal accretive operators.

Theorem 3.2: *Let Φ, Ψ be a s.n. functions, $p \geqslant 2$, $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that A, B are accretive. If $AX + XB \in \mathfrak{C}_\Psi(\mathcal{H})$ then also $\sqrt{A^* + A}X\sqrt{B + B^*} \in \mathfrak{C}_\Psi(\mathcal{H})$ and*

$$\sqrt{A^* + A}X\sqrt{B + B^*} = \int_{\mathbb{R}_+} \sqrt{A^* + A}\, \mathrm{e}^{-tA}(AX + XB)\, \mathrm{e}^{-tB} \sqrt{B + B^*}\, \mathrm{d}t, \tag{32}$$

$$\boxed{\left\| \sqrt{A^* + A}X\sqrt{B + B^*} \right\|_\Psi \leqslant \|AX + XB\|_\Psi,} \tag{33}$$

hold under any of the following conditions:

(a1) $\Psi := \Phi^{(p)*}$ *and A is quasinormal having the adjoint operator which is 2-hyperaccretive operator with injective real part (in which case* ${}^{s}\lim_{t \to +\infty} \sqrt{A^* + A}\, \mathrm{e}^{-tA} = 0$, *as well as* ${}^{\mathfrak{C}_2(\mathcal{H})}\lim_{t \to +\infty} \sqrt{A^* + A}\, \mathrm{e}^{-tA} X\, \mathrm{e}^{-tB} \sqrt{B + B^*} = 0$*);*

(a2) $\Psi := \Phi^{(p)*}$ *and B^* is quasinormal having the adjoint operator which is 2-hyperaccretive operator with injective real part (in which case* ${}^{s}\lim_{t \to +\infty} \mathrm{e}^{-tB} \sqrt{B + B^*} = 0$, *as well as* ${}^{\mathfrak{C}_2(\mathcal{H})}\lim_{t \to +\infty} \sqrt{A^* + A}\, \mathrm{e}^{-tA} X\, \mathrm{e}^{-tB} \sqrt{B + B^*} = 0$*);*

(b) *if both A and B^* are quasinormal operators having its adjoint operators which are 2-hyperaccretive operators with injective real parts.*

(c1) *If $AX + XB \in \mathfrak{C}^{(\circ)}_{\Phi^{(p)}}(\mathcal{H})$, A is quasinormal operator having the adjoint operator which is 2-hyperaccretive operator with injective real part and B is strictly accretive, then*

$$\sqrt{A^* + A}\,X = \int_{\mathbb{R}_+} \sqrt{A^* + A}\,\mathrm{e}^{-tA}(AX + XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t, \tag{34}$$

$$\left\|\sqrt{A^* + A}\,X\right\|_{\Phi^{(p)}} \leqslant \|(AX + XB)(B + B^*)^{-1/2}\|_{\Phi^{(p)}}. \tag{35}$$

(c2) *If $AX + XB \in \mathfrak{C}^{(\circ)}_{\Phi^{(p)}}(\mathcal{H})$, A is strictly accretive and B^* is quasinormal operator having the adjoint operator which is 2-hyperaccretive operator with injective real part, then*

$$X\sqrt{B + B^*} = \int_{\mathbb{R}_+} \mathrm{e}^{-tA}(AX + XB)\,\mathrm{e}^{-tB}\sqrt{B + B^*}\,\mathrm{d}t, \tag{36}$$

$$\left\|X\sqrt{B + B^*}\right\|_{\Phi^{(p)}} \leqslant \|(A^* + A)^{-1/2}(AX + XB)\|_{\Phi^{(p)}}. \tag{37}$$

Proof: To prove (a1), let us note that

$$\begin{aligned}
&{}^{s}\!\lim_{t\to+\infty}\left|\begin{bmatrix}\sqrt{A^*+A}\,\mathrm{e}^{-tA} & * \\ 0 & *\end{bmatrix}\right|^2 \\
&:= {}^{s}\!\lim_{t\to+\infty}\left|\begin{bmatrix}\sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I - C_A^2}\sqrt{A^*+A}\end{bmatrix}\begin{bmatrix}\mathrm{e}^{-tA} & V_t \\ 0 & \mathrm{e}^{-tA^*}\end{bmatrix}\right|^2 \\
&= {}^{s}\!\lim_{t\to+\infty}\left|\begin{bmatrix}\sqrt{A^*+A} & C_A\sqrt{A^*+A} \\ 0 & \sqrt{I - C_A^2}\sqrt{A^*+A}\end{bmatrix}\mathrm{e}^{-t\tilde{A}}\right|^2 = {}^{s}\!\lim_{t\to+\infty}\mathrm{e}^{-t\tilde{A}^*}\left(\tilde{A}^* + \tilde{A}\right)\mathrm{e}^{-t\tilde{A}} \\
&= {}^{s}\!\lim_{t\to+\infty}\left|\sqrt{\tilde{A}^* + \tilde{A}}\,\mathrm{e}^{-t\tilde{A}}\right|^2 = 0,
\end{aligned} \tag{38}$$

according to [9, Lemma 2.5(e)] as $\tilde{A}$ is 2-hyperaccretive due to Example 2.3. This allows us to conclude that ${}^{s}\!\lim_{t\to+\infty}\sqrt{A^*+A}\,\mathrm{e}^{-tA} = 0$ as well, based on (b) and (c) in Lemma 2.1. By applying the last equality in (38) to the normal operator $\widetilde{B^*}$ instead of $\tilde{A}$ in the case (a2) it follows that $\lim_{t\to+\infty}\left\|\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*} + \widetilde{B^*}^*}\tilde{h}\right\| = \lim_{t\to+\infty}\left\|\sqrt{\widetilde{B^*}^* + \widetilde{B^*}}\,\mathrm{e}^{-t\widetilde{B^*}}\tilde{h}\right\| = 0$ for all $\tilde{h} \in \mathcal{H} \oplus \mathcal{H}$. Thus, for every $h \in \mathcal{H}$ and $\tilde{h} := [0, h]^\top$ a straightforward calculation shows

$$\begin{aligned}
&\lim_{t\to+\infty}\left\|\mathrm{e}^{-tB}\sqrt{B+B^*}h\right\|^2 = \lim_{t\to+\infty}\left\|\begin{bmatrix}0 & \mathrm{e}^{-tB}\sqrt{B+B^*} \\ 0 & 0\end{bmatrix}\tilde{h}\right\|^2 \\
&= \lim_{t\to+\infty}\left\|\begin{bmatrix}0 & I \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{e}^{-tB^*} & W_t \\ 0 & \mathrm{e}^{-tB}\end{bmatrix}\begin{bmatrix}\sqrt{B+B^*}\sqrt{I - C_{B^*}^2} & \sqrt{B+B^*}C_{B^*} \\ 0 & \sqrt{B+B^*}\end{bmatrix}\tilde{h}\right\|^2 \\
&= \lim_{t\to+\infty}\left\|\check{I}\,\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*} + \widetilde{B^*}^*}\,U_{\widehat{B^*}}{}^*\tilde{h}\right\|^2 = 0,
\end{aligned}$$

i.e. ${}^{s}\lim_{t\to+\infty} \mathrm{e}^{-tB}\sqrt{B+B^*} = 0$. As $\sup_{t\geqslant 0} \left\|\sqrt{A^*+A}\,\mathrm{e}^{-tA}X\right\| \leqslant \|A^*+A\|^{1/2}\|X\| < +\infty$, then

$$ {}^{s}\lim_{t\to+\infty} \sqrt{A^*+A}\,\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\sqrt{B+B^*} = 0. \tag{39} $$

The Newton-Leibnitz formula (2) implies

$$ \begin{aligned} &\sqrt{A^*+A}X\sqrt{B+B^*} - \sqrt{A^*+A}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{B+B^*} \\ &\quad = \int_{[0,T]} \sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t \quad \text{for all } T>0. \end{aligned} \tag{40} $$

The existence of $V^{-}_{A,B}X$ as ${}^{\mathfrak{C}_2(\mathcal{H})}\lim_{t\to+\infty} \sqrt{A^*+A}\,\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\sqrt{B+B^*}$ in the case (a2) follows from the following estimate for all $0<T<S$

$$ \begin{aligned} &\left\|\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{B+B^*} - \sqrt{A^*+A}\,\mathrm{e}^{-SA}X\,\mathrm{e}^{-SB}\sqrt{B+B^*}\right\|_2 \\ &\quad = \left\|\int_{(T,S]} \sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t\right\|_2 \\ &\quad = \left\|\int_{(T,S]} \begin{bmatrix}\sqrt{A^*+A} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{e}^{-tA} & 0\\ 0 & 0\end{bmatrix}\left(\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & X\\ 0 & 0\end{bmatrix}\right.\right. \\ &\qquad \left. + \begin{bmatrix}0 & X\\ 0 & 0\end{bmatrix}\begin{bmatrix}B^* & \sqrt{BB^*-B^*B}\\ 0 & B\end{bmatrix}\right) \\ &\qquad \left. \times \begin{bmatrix}\mathrm{e}^{-tB^*} & W_t\\ 0 & \mathrm{e}^{-tB}\end{bmatrix}\begin{bmatrix}\sqrt{B+B^*}\sqrt{I-C^2_{B^*}} & \sqrt{B+B^*}C_{B^*}\\ 0 & \sqrt{B+B^*}\end{bmatrix}\mathrm{d}t\right\|_2 \\ &\quad = \left\|\int_{(T,S]} \begin{bmatrix}\sqrt{A^*+A} & 0\\ 0 & 0\end{bmatrix}\mathrm{e}^{-tA\oplus 0}((A\oplus 0)\check{X} + \check{X}\widetilde{B^*})\,\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*}+\widetilde{B^*}^*}\,U_{\widehat{B^*}}{}^*\,\mathrm{d}t\right\|_2 \\ &\quad \leqslant \left\|\int_{(T,+\infty)} (\sqrt{A^*+A}\,\mathrm{e}^{-tA}\oplus 0)((A\oplus 0)\check{X} + \check{X}\widetilde{B^*})\,\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*}+\widetilde{B^*}^*}\,\mathrm{d}t\right\|_2 \\ &\qquad + \left\|\int_{(S,+\infty)} (\sqrt{A^*+A}\,\mathrm{e}^{-tA}\oplus 0)((A\oplus 0)\check{X} + \check{X}\widetilde{B^*})\,\mathrm{e}^{-t\widetilde{B^*}}\sqrt{\widetilde{B^*}+\widetilde{B^*}^*}\,\mathrm{d}t\right\|_2 \to 0 \end{aligned} \tag{41} $$

as $T\to+\infty$, according to [5, Th. 3.1(d)], applied to s.i. family $\left\{\sqrt{A^*+A}\,\mathrm{e}^{-tA}\oplus 0\right\}_{t>T}$ and s.i. and m.c.n.o. family $\left\{\left(\widetilde{B^*}+\widetilde{B^*}^*\right)^{1/2}\mathrm{e}^{-t\widetilde{B^*}^*}\right\}_{t>T}$, where $\widetilde{B^*} := \left[\begin{smallmatrix}B^* & \sqrt{BB^*-B^*B}\\ 0 & B\end{smallmatrix}\right]$. Arguments supporting equalities in (41) are already used in the proof of the equalities (21)–(22). The existence of ${}^{\mathfrak{C}_2(\mathcal{H})}\lim_{T\to+\infty} \int_{[0,T]} \sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t$ follows from the estimate (41) as well, and moreover, according to the formula (40) it equals to ${}^{\mathfrak{C}_2(\mathcal{H})}\lim_{T\to+\infty} \sqrt{A^*+A}\left(X - \mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\right)\sqrt{B+B^*}$, which further equals to $\sqrt{A^*+A}X\sqrt{B+B^*}$ as revealed by the formula (39). This proves the formula (32) in the case (a2) and moreover, this also shows that $\sqrt{A^*+A}X\sqrt{B+B^*} \in \mathfrak{C}_2(\mathcal{H})$.

To prove formula (32) in the case (a1), we use the adjoint property of the Hilbert-Schmidt norm $\|\cdot\|_2$ to get

$$\lim_{T\to+\infty}\left\|\int_{(T,+\infty)}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t\right\|_2$$
$$=\lim_{T\to+\infty}\left\|\int_{(T,+\infty)}\sqrt{B+B^*}\,\mathrm{e}^{-tB^*}(AX+XB)^*\,\mathrm{e}^{-tA^*}\sqrt{A^*+A}\,\mathrm{d}t\right\|_2=0,$$

according to the already proven part (a2), applied to (B^*, A^*, X^*) instead of (A, B, X). So by the formula (40) it follows

$$\begin{aligned}&{}^{\mathfrak{e}_2(\mathcal{H})}\lim_{T\to+\infty}\sqrt{A^*+A}\left(X-\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\right)\sqrt{B+B^*}\\&=\int_{\mathbb{R}_+}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t.\end{aligned}\tag{42}$$

By applying again the already proven part (a2) to (B^*, A^*, X^*) instead of (A, B, X), we conclude that ${}^s\lim_{t\to+\infty}\sqrt{B+B^*}\,\mathrm{e}^{-tB^*}X^*\mathrm{e}^{-tA^*}\sqrt{A^*+A}=0$, which therefore implies ${}^w\lim_{t\to+\infty}\sqrt{A^*+A}\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\sqrt{B+B^*}=0$. This allows us to recognize that ${}^{\mathfrak{e}_2(\mathcal{H})}\lim_{t\to+\infty}\sqrt{A^*+A}\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\sqrt{B+B^*}=0$, which by the formula (42) proves (32).

Based on the polar representations (25) and (26), we get

$$\begin{aligned}&\begin{bmatrix}0 & \int_{(T,+\infty)}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t\\ 0 & 0\end{bmatrix}\\
&=\int_{(T,+\infty)}\begin{bmatrix}\sqrt{A^*+A} & C_A\sqrt{A^*+A}\\ 0 & \sqrt{I-C_A^2}\sqrt{A+A^*}\end{bmatrix}\\
&\times\mathrm{e}^{-t\tilde{A}}(\widetilde{A}\check{X}+\check{X}\widetilde{B})\,\mathrm{e}^{-t\widetilde{B^*}^*}\begin{bmatrix}\sqrt{B+B^*}\sqrt{I-C_{B^*}^2} & \sqrt{B+B^*}C_{B^*}\\ 0 & \sqrt{B+B^*}\end{bmatrix}\mathrm{d}t\\
&=\int_{(T,+\infty)}U_{\check{A}}\sqrt{\tilde{A}^*+\tilde{A}}\,\mathrm{e}^{-t\tilde{A}}(\widetilde{A}\check{X}+\check{X}\widetilde{B})\,\mathrm{e}^{-t\widetilde{B^*}^*}\sqrt{\widetilde{B^*}^*+\widetilde{B^*}}\,U_{\widehat{B^*}}{}^*\,\mathrm{d}t\xrightarrow{\ s\ }0\quad\text{as } T\to+\infty,\end{aligned}$$

according to [5, Lemma 2.1(b)], applied to $\widetilde{A}\check{X}+\check{X}\widetilde{B}$ instead of X, m.c.n.o. and s.i. families $\{A_t\}_{t\in\mathbb{R}_+}$ and $\{B_t\}_{t\in\mathbb{R}_+}$ defined by $A_t:=\sqrt{\tilde{A}^*+\tilde{A}}\,\mathrm{e}^{-t\tilde{A}}$, $B_t:=\mathrm{e}^{-t\widetilde{B^*}^*}\sqrt{\widetilde{B^*}^*+\widetilde{B^*}}$, $\delta:=\Omega:=\mathbb{R}_+$ and $\delta_n:=[0,T_n]$ for any increasing sequence $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}T_n=+\infty$. Thus ${}^s\lim_{T\to+\infty}\int_{(T,+\infty)}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t=0$ as well, i.e.

$$\begin{aligned}&\int_{\mathbb{R}_+}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t\\
&={}^s\lim_{T\to+\infty}\int_{[0,T]}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\sqrt{B+B^*}\,\mathrm{d}t\\
&={}^s\lim_{T\to+\infty}\left(\sqrt{A^*+A}X\sqrt{B+B^*}-\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{B+B^*}\right)\\
&=\sqrt{A^*+A}X\sqrt{B+B^*},\end{aligned}\tag{43}$$

which actually proves formula (32) in the case (b). Here, the second equality in (43) is based on the formula (40), while the last equality in (43) follows from the already proven facts that ${}^{s}\lim_{T\to+\infty} \mathrm{e}^{-TB}\sqrt{B+B^*}=0$ in (a2) and $\sup_{T\geqslant 0}\left\|\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\right\|<+\infty$.

A straightforward adaptation of the proof of the equalities in (27) provides

$$\begin{aligned}
&\left\|\int_{[T,+\infty)}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t\right\| \\
&\quad\leqslant \left\|\int_{[T,+\infty)}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t\right\|_{\Phi^{(p)}} \\
&\quad= \left\|U_{\tilde{A}}\int_{[T,+\infty)}\sqrt{\widetilde{A}+\widetilde{A}^*}\,\mathrm{e}^{-t\tilde{A}}\begin{bmatrix}0 & (AX+XB)(B+B^*)^{-\frac{1}{2}}\\ 0 & 0\end{bmatrix}\left(0\oplus\sqrt{B^*+B}\,\mathrm{e}^{-tB}\right)\,\mathrm{d}t\right\|_{\Phi^{(p)}} \\
&\quad\to 0
\end{aligned}$$

as $T\to+\infty$, according to the convergence property in $\mathcal{C}_{\Phi^{(p)}}(\mathcal{H})$ presented in [5, Th. 3.1(a)], applied to $\left[\begin{smallmatrix}0 & (AX+XB)(B+B^*)^{-1/2}\\ 0 & 0\end{smallmatrix}\right]\in\mathcal{C}^{(\circ)}_{\Phi^{(p)}}(\mathcal{H}\oplus\mathcal{H})$, to s.i. and m.c.n.o. family $\boldsymbol{A}:=\left\{\left(\widetilde{A}^*+\widetilde{A}\right)^{1/2}\mathrm{e}^{-t\tilde{A}}\right\}_{t>T}$ and to s.i. family $\boldsymbol{B}:=\left\{0\oplus\sqrt{B^*+B}\,\mathrm{e}^{-tB}\right\}_{t>T}$. Consequently, ${}^{s}\lim_{T\to+\infty}\int_{(T,+\infty)}\sqrt{A^*+A}\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t=0$, implying that

$$\begin{aligned}
\int_{\mathbb{R}_+}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t &= {}^{s}\!\lim_{T\to+\infty}\int_{[0,T]}\sqrt{A^*+A}\,\mathrm{e}^{-tA}(AX+XB)\,\mathrm{e}^{-tB}\,\mathrm{d}t \\
&= {}^{s}\!\lim_{T\to+\infty}\left(\sqrt{A^*+A}X-\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\right)=\sqrt{A^*+A}X, \qquad (44)
\end{aligned}$$

which actually proves formula (34) in the case (c1). Indeed, the second equality in (44) is based on the formula (2), while the last equality in (44) follows from the facts that ${}^{s}\lim_{T\to+\infty}\mathrm{e}^{-TB}=0$ by Jocić et al. [9, Lemma 2.5(f)] and $\sup_{T\geqslant 0}\left\|\sqrt{A^*+A}\,\mathrm{e}^{-TA}X\right\|<+\infty$.

The formula (36) in the case (c2) proves similarly as the formula (34) in (c1), supported by the fact that ${}^{s}\lim_{T\to+\infty}\mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\sqrt{B+B^*}=0$ as ${}^{s}\lim_{T\to+\infty}\mathrm{e}^{-TB}\sqrt{B+B^*}=0$ by (a2) and $\sup_{T\geqslant 0}\|\mathrm{e}^{-TA}X\|<+\infty$.

To prove the formula (33), we just have to use the expansion formula (32) and then apply the inequality (18) to $AX+XB$ instead to X.

For the proof of the formula (35) (resp. (37)) we use the expansion formula (34) (resp. (36)) to apply the inequality (19) (resp. (20)) to $AX+XB$ instead to X. ■

3.2. Higher order derivatives $(A\otimes I+I\otimes B)^N$ and its related transformers

Lemma 3.3: *If $A,B,X\in\mathcal{B}(\mathcal{H})$ then ${}^{\mathcal{B}(\mathcal{H})}\lim_{t\to 0}\frac{1}{t}\left(X-\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\right)=AX+XB$. If, in addition, $A^2X+2AXB+XB^2\in\mathcal{C}_\Psi(\mathcal{H})$ for some s.n. function Ψ, then*

$$ {}^{\Psi}\!\lim_{t\to 0}\frac{1}{t}\left(X-\mathrm{e}^{-tA}X\,\mathrm{e}^{-tB}\right)=AX+XB, \qquad (45)$$

$$\lim_{t\to 0}\left\|\frac{1}{t}\left(I\otimes I-\mathrm{e}^{-tA}\otimes\mathrm{e}^{-tB}\right)-A\otimes I-I\otimes B\right\|_{\mathcal{B}(\mathcal{C}_\Psi(\mathcal{H}))}=0. \qquad (46)$$

Moreover, A is contractive if and only if e^{-tA} *is accretive for all* $t \geqslant 0$.

Proof: The formula (45) follows from the estimate

$$
\begin{aligned}
&\left\| \frac{1}{t}(X - \mathrm{e}^{-tA} X \,\mathrm{e}^{-tB}) - AX - XB \right\|_\Psi \\
&\quad = \frac{1}{t} \left\| \int_0^t \left(\mathrm{e}^{-sA}(AX + XB)\,\mathrm{e}^{-sB} - AX - XB \right) \,\mathrm{d}s \right\|_\Psi \\
&\quad = \frac{1}{t} \left\| \int_0^t \int_0^s \mathrm{e}^{-uA}(A^2X + 2AXB + XB^2)\,\mathrm{e}^{-uB} \,\mathrm{d}u \,\mathrm{d}s \right\|_\Psi && (47)\\
&\quad = \frac{1}{t} \left\| \int_0^t (t-u)\,\mathrm{e}^{-uA}(A^2X + 2AXB + XB^2)\,\mathrm{e}^{-uB} \,\mathrm{d}u \right\|_\Psi && (48)\\
&\quad \leqslant \frac{1}{t} \int_0^t (t-u) \|\mathrm{e}^{-uA}(A^2X + 2AXB + XB^2)\,\mathrm{e}^{-uB}\|_\Psi \,\mathrm{d}u \\
&\quad \leqslant \frac{1}{t} \|A^2X + 2AXB + XB^2\|_\Psi \int_0^t (t-u)\,\mathrm{e}^{u(\|A\|+\|B\|)} \,\mathrm{d}u \\
&\quad = t\frac{\mathrm{e}^{t(\|A\|+\|B\|)} - 1 - t(\|A\| + \|B\|)}{t^2(\|A\| + \|B\|)^2} \|A^2X + 2AXB + XB^2\|_\Psi \to 0, \quad \text{as } t \searrow 0. && (49)
\end{aligned}
$$

Both equalities in (47) are due to Newton–Leibnitz formula (2), while the identity in (48) follows by the Fubini theorem for $\mathfrak{B}(\mathcal{H})$ functions, and the remaining inequalities and the equality are based on elementary calculations. The case $t \nearrow 0$ derives by applying the already proven case $t \searrow 0$ to $(-A, -B)$ instead of (A, B). The proof for

$$
{}^{\mathfrak{B}(\mathcal{H})}\lim_{t\to 0} \frac{1}{t} \left(X - \mathrm{e}^{-tA} X \,\mathrm{e}^{-tB} \right) = AX + XB \tag{50}
$$

goes similarly to the proof of the formula (45).

As $\left\| \frac{I\otimes I - \mathrm{e}^{-tA} \otimes \mathrm{e}^{-tB}}{t} - A \otimes I - I \otimes B \right\|_{\mathfrak{B}(\mathfrak{C}_\Psi(\mathcal{H}))} \leqslant |t| \frac{\mathrm{e}^{|t|(\|A\|+\|B\|)} - 1 - |t|(\|A\|+\|B\|)}{t^2}$ for all $t \in \mathbb{R} \setminus \{0\}$ based on the estimate (49), the formula (46) follows.

A special case of formula (9) in Ref. [9] says that $I - \mathrm{e}^{-tA^*}\mathrm{e}^{-tA} = \int_0^t \mathrm{e}^{-sA^*}(A^* + A)\,\mathrm{e}^{-sA} \geqslant 0$ for all $t \geqslant 0$ if $A^* + A \geqslant 0$, i.e. e^{-tA} is contraction if A is accretive. If e^{-tA} is contraction for all $t \geqslant 0$ then the formula (50) implies $A^* + A = {}^{\mathfrak{B}(\mathcal{H})}\lim_{t\searrow 0} \frac{I - \mathrm{e}^{-tA^*}\mathrm{e}^{-tA}}{t} \geqslant 0$. ■

Further generalization of Lemma 3.3 and some relations between N-hyperaccretive and N-hypercontractive operators are presented in the following lemma.

Lemma 3.4: *Let Ψ be a s.n. function and let $A, B, X \in \mathfrak{B}(\mathcal{H})$. Then*

(a)

$$
\Gamma^K_{\mathrm{e}^{-tA},\mathrm{e}^{-tB}}(X) = \int_{[0,t]^K} \mathrm{e}^{-\sum_{k=1}^K t_k A} \left(\sum_{k=0}^K \binom{K}{k} A^{K-k} X B^k \right) \mathrm{e}^{-\sum_{k=1}^K t_k B} \,\mathrm{d}t_1 \cdots \mathrm{d}t_K, \tag{51}
$$

$$\Gamma^K_{e^{-tB^*},e^{-tB}}(I) = \int_{[0,t]^K} e^{-\sum_{k=1}^K t_k B^*} \left(\sum_{k=0}^K \binom{K}{k} B^{*k} B^{K-k} \right) e^{-\sum_{k=1}^K t_k B} \, dt_1 \cdots dt_K. \tag{52}$$

$$\lim_{t\searrow 0} \left\| t^{-K} \Gamma^K_{e^{-tA^*},e^{-tA}}(I) - \sum_{k=0}^K \binom{K}{k} A^{*k} A^{K-k} \right\| = 0. \tag{53}$$

If, in addition, $\sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^k \in \mathfrak{C}_\Psi(\mathcal{H})$ *for some* $K \in \mathbb{N}$*, then for all* $t > 0$

$$\lim_{t\searrow 0} \left\| t^{-K} \Gamma^K_{e^{-tA},e^{-tB}}(X) - \sum_{k=0}^K \binom{K}{k} A^{K-k} X B^k \right\|_\Psi = 0. \tag{54}$$

(b) *Operator* $B \in \mathfrak{B}(\mathcal{H})$ *is* K*-hyperaccretive if and only if* e^{-tB} *is* K*-hypercontractive for all* $t > 0$.

(c) *If* e^{-tB} *is quasinormal for all* $t > 0$*, then* B *is normal.*

(d) *A contraction* $C \in \mathfrak{B}(\mathcal{H})$ *is* K*-hypercontractive if and only if Cayley transforms* $\mathcal{C}_{rC} \stackrel{\text{def}}{=} (I - rC)(I + rC)^{-1}$ *are* K*-hyperaccretive for all* $r \in (0, 1)$*. Moreover, an accretive operator* $A \in \mathfrak{B}(\mathcal{H})$ *is* K*-hyperaccretive if and only if for all* $\varepsilon > 0$ *Cayley transforms* $\mathcal{C}_{A+\varepsilon I} \stackrel{\text{def}}{=} (I - A - \varepsilon I)(I + A + \varepsilon I)^{-1}$ *are* K*-hypercontractive.*

Proof: Lemma 3.3 shows the validity of the case $K := 1$ in the part (a) of this lemma. Also, as $\Gamma_{e^{-tA},e^{-tB}}(X) = X - e^{-tA} X e^{-tB} = \int_0^t e^{-sA}(AX + XB) e^{-sB} ds$, the formula (51) follows directly by induction. Formula (52) is a special case $(A, X) := (B^*, I)$ of the formula (51).

To prove (54), we rely on the estimate

$$\left\| t^{-K} \Gamma^K_{e^{-tA},e^{-tB}}(X) - \sum_{k=0}^K \binom{K}{k} A^{K-k} X B^k \right\|_\Psi \tag{55}$$

$$= t^{-K} \left\| \int_{[0,t]^K} \left(e^{-\sum_{k=1}^K t_k A} \left(\sum_{k=0}^K \binom{K}{k} A^{K-k} X B^k \right) e^{-\sum_{k=1}^K t_k B} - \sum_{k=0}^K \binom{K}{k} A^{K-k} X B^k \right) dt_1 \cdots dt_K \right\|_\Psi$$

$$= t^{-K} \left\| \int_{[0,t]^K} \int_{[0,\sum_{k=1}^K t_k]} e^{-sA} \left(\sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^k \right) e^{-sB} \, ds \, dt_1 \cdots dt_K \right\|_\Psi$$

$$\leqslant t^{-K} \int_{[0,t]^K} \int_{[0,\sum_{k=1}^K t_k]} e^{s\|A\|} \left\| \sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^k \right\|_\Psi e^{s\|B\|} \, ds \, dt_1 \cdots dt_K$$

$$\leqslant t^{-K} e^{Kt(\|A\|+\|B\|)} \left\| \sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^k \right\|_\Psi \int_{[0,t]^K} \sum_{k=1}^K t_k \, dt_1 \cdots dt_K$$

$$= \frac{Kt}{2} e^{Kt(\|A\|+\|B\|)} \left\| \sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^k \right\|_\Psi \to 0 \qquad \text{as } t \searrow 0. \tag{56}$$

The equality (55) is due to the formula (51) and the equality (56) follows from the application of the formula (2) to $\mathrm{T} := \sum_{k=1}^{K} t_k$ and to $\sum_{k=0}^{K} \binom{K}{k} A^{K-k} X B^k$ instead of X. The proof of the formula (54) shows that this formula remains valid for arbitrary $A, B, X \in \mathcal{B}(\mathcal{H})$ if Ψ is replaced with $\mathcal{B}(\mathcal{H})$, so the special choice (A^*, A, I) for (A, B, X) gives the formula (53).

To prove (b) for N-hyperaccretive B, we use the formula (52) to see that $\Gamma^K_{e^{-tB^*}, e^{-tB}}(I) \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$, and so e^{-tB} is N-hypercontractive. If e^{-tB} is N-hypercontractive for all $t > 0$, then the formula (53) applied to $A := B$ implies that $\sum_{k=0}^{K} \binom{K}{k} B^{*k} B^{K-k} \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$, showing that B is N-hyperaccretive.

If e^{-tB} is quasinormal for all $t > 0$ in the case (c), then by definition $(e^{-tB^*} e^{-tB} - e^{-tB} e^{-tB^*})\, e^{-tB} = 0$, and consequently $e^{-tB^*} e^{-tB} = e^{-tB} e^{-tB^*}$ for all $t > 0$. A double consequent application of the formula (45) shows that

$$
\begin{aligned}
e^{-tB^*}(B^{*2} + 2B^*B + B^2)\, e^{-tB} &= \frac{d^2}{dt^2}\left(e^{-tB^*} e^{-tB}\right) \\
&= \frac{d^2}{dt^2}\left(e^{-tB} e^{-tB^*}\right) = e^{-tB}(B^2 + 2BB^* + B^{*2})\, e^{-tB^*},
\end{aligned}
$$

so by letting $t \searrow 0$ we conclude $B^*B = BB^*$, as stated in (c).

To prove (d) let us note that C is N-hypercontractive if and only if rC is N-hypercontractive for all $r \in (0,1)$ according to [11, Lemma 2.15] and the fact that $(I - C^* \otimes C)^K(I) = {}^{\mathcal{B}(\mathcal{H})}\lim_{r \nearrow 1}(I - rC^* \otimes rC)^K(I) \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$. If $I + D$ is (boundedly) invertible for some $D \in \mathcal{B}(\mathcal{H})$ (i.e. if $-1 \notin \sigma(D)$), then

$$
\begin{aligned}
\sum_{k=0}^{K} \tbinom{K}{k} \mathcal{C}_D^{*k} \mathcal{C}_D^{K-k} &= \sum_{k=0}^{K} \tbinom{K}{k} (I - D^*)^k (I + D^*)^{-k} (I - D)^{K-k} (I + D)^{k-K} \\
&= (I + D^*)^{-K} \sum_{k=0}^{K} \tbinom{K}{k} (I - D^*)^k (I + D^*)^{K-k} (I - D)^{K-k} (I + D)^k (I + D)^{-K} \\
&= (I + D^*)^{-K} \sum_{k=0}^{K} \tbinom{K}{k} \left(\left((I - D^*) \otimes (I + D)\right)^k \left((I + D^*) \otimes (I - D)\right)^{K-k}\right)(I)(I + D)^{-K} \\
&= (I + D^*)^{-K} \left((I - D^*) \otimes (I + D) + (I + D^*) \otimes (I - D)\right)^K (I)(I + D)^{-K} \\
&= 2^K (I + D^*)^{-K} (I \otimes I - D^* \otimes D)^K (I)(I + D)^{-K}. \qquad (57)
\end{aligned}
$$

By applying (57) to rC instead of D we get

$$
\sum_{k=0}^{K} \tbinom{K}{k} \mathcal{C}_{rC}^{*k} \mathcal{C}_{rC}^{K-k} = 2^K (I + rC^*)^{-K} (I \otimes I - r^2 C^* \otimes C)^K (I)(I + rC)^{-K} \geqslant 0. \qquad (58)
$$

The equality in (58) shows that $\mathcal{C}_{rC}$ is N-hyperaccretive for all $r \in (0,1)$ if and only if rC is N-hypercontractive for all $r \in (0,1)$, which was shown to be equivalent to the N-hypercontractivity of C.

To prove the last statement in (d), let A be accretive and $\varepsilon > 0$, in which case $A_\varepsilon := A + \varepsilon I$ is also accretive and it satisfies $(I + \mathcal{C}_{A_\varepsilon})^{-1} = \left(I + (I - A - \varepsilon I)(I + A + \varepsilon I)^{-1}\right)^{-1} = (I + A + \varepsilon I)/2 \in \mathcal{B}(\mathcal{H})$, so by applying (57) to $\mathcal{C}_{A_\varepsilon}$ instead of D it follows

$$
2^K (I + \mathcal{C}_{A_\varepsilon}^*)^{-K} (I \otimes I - \mathcal{C}_{A_\varepsilon}^* \otimes \mathcal{C}_{A_\varepsilon})^K (I)(I + \mathcal{C}_{A_\varepsilon})^{-K}
$$

$$= \sum_{k=0}^{K} \binom{K}{k} \mathcal{C}_{\mathcal{C}_{A_\varepsilon}}^{*k} \mathcal{C}_{\mathcal{C}_{A_\varepsilon}}^{K-k} = \sum_{k=0}^{K} \binom{K}{k} A_\varepsilon^{*k} A_\varepsilon^{K-k}.$$

Therefore $(I \otimes I - \mathcal{C}_{A_\varepsilon}^* \otimes \mathcal{C}_{A_\varepsilon})^K(I) \geqslant 0$ for all $\mathbb{N} \ni K \leqslant N$, i.e. $\mathcal{C}_{A_\varepsilon}$ is N-hypercontractive for all $\varepsilon > 0$ if and only if A_ε is N-hyperaccretive for all $\varepsilon > 0$, so by Lemma 2.6 this is the case if and only if A is N-hyperaccretive.

Note that if any of those equivalent conditions are satisfied, then

$$\sum_{k=0}^{K} (-1)^k \binom{K}{k} \mathcal{C}_A^{*k} \mathcal{C}_A^k = \sum_{k=0}^{K} (-1)^k \binom{K}{k} (I - A^*)^k (I + A^*)^{-k} (I - A)^k (I + A)^{-k}$$

$$= {}^{\mathcal{B}(\mathcal{H})}\lim_{\varepsilon \searrow 0} \sum_{k=0}^{K} (-1)^k \binom{K}{k} (I - A_\varepsilon^*)^k (I + A_\varepsilon^*)^{-k} (I - A_\varepsilon)^k (I + A_\varepsilon)^{-k}$$

$$= {}^{\mathcal{B}(\mathcal{H})}\lim_{\varepsilon \searrow 0} \sum_{k=0}^{K} \binom{K}{k} A_\varepsilon^{*k} A_\varepsilon^{K-k} \geqslant 0,$$

which shows that $\mathcal{C}_A$ is also N-hypercontractive. ■

In the sequel, the following generalization of Lemma 3.3 will be needed.

Lemma 3.5: *If $\{c_k\}_{k=1}^n$ is a sequence in $\mathbb{R}_+$ for some $n \in \mathbb{N}$, $A, B \in \mathfrak{B}(\mathcal{H})$ and $X \in \mathfrak{C}_\Psi(\mathcal{H})$, then*

$$\lim_{t \searrow 0} \left\| \frac{1}{t^n} \prod_{k=1}^{n} \left(I \otimes I - \mathrm{e}^{-t(c_k + A)} \otimes \mathrm{e}^{-tB} \right) - \prod_{k=1}^{n} \left((c_k I + A) \otimes I + I \otimes B \right) \right\|_{\mathfrak{B}(\mathfrak{C}_\Psi(\mathcal{H}))} = 0, \tag{59}$$

$${}^{\Psi}\lim_{t \searrow 0} \frac{1}{t^n} \prod_{k=1}^{n} \left(I \otimes I - \mathrm{e}^{-t(c_k + A)} \otimes \mathrm{e}^{-tB} \right) (X) = \prod_{k=1}^{n} \left((c_k I + A) \otimes I + I \otimes B \right) (X). \tag{60}$$

Proof: As ${}^{\mathfrak{B}(\mathfrak{C}_\Psi(\mathcal{H}))}\lim_{t \searrow 0} \frac{1}{t} \left(I \otimes I - \mathrm{e}^{-t(c_k + A)} \otimes \mathrm{e}^{-tB} \right) = (c_k I + A) \otimes I + I \otimes B$ for all $\mathbb{N} \ni k \leqslant n$ based on the multiple application of the formula (46) in Lemma 3.3, the consequent application of the formula (4) provides the formula (59). Thus, the formula (60) follows directly from the formula (59). ■

Theorem 3.6: *Let Φ, Ψ be s.n. functions, $J, K, M, N \in \mathbb{N}_\circ$ satisfy $K \leqslant M \leqslant J \leqslant N$, let $A, B, X \in \mathfrak{B}(\mathcal{H})$ be such that A and B^* are N-hyperaccretive and $X \in \mathfrak{C}_\Psi(\mathcal{H})$. Then*

$$\left\| \left(\sum_{i=0}^{J-K} \binom{J-K}{i} A^{*i} A^{J-K-i} \right)^{1/2} \sum_{k=0}^{K} \binom{K}{k} A^{K-k} X B^k \left(\sum_{i=0}^{J-K} \binom{J-K}{i} B^{J-K-i} B^{*i} \right)^{1/2} \right\|_\Psi$$

$$\leqslant \left\| \left(\sum_{i=0}^{J-M} \binom{J-M}{i} A^{*i} A^{J-M-i} \right)^{1/2} \sum_{m=0}^{M} \binom{M}{m} A^{M-m} X B^m \left(\sum_{i=0}^{J-M} \binom{J-M}{i} B^{J-M-i} B^{*i} \right)^{1/2} \right\|_\Psi, \tag{61}$$

and specially for $K := 0$ *and* $J := M := N$

$$\left\| \left(\sum_{n=0}^{N} \binom{N}{n} A^{*n} A^{N-n} \right)^{1/2} X \left(\sum_{n=0}^{N} \binom{N}{n} B^{N-n} B^{*n} \right)^{1/2} \right\|_{\Psi} \leqslant \left\| \sum_{n=0}^{N} \binom{N}{n} A^{N-n} X B^{n} \right\|_{\Psi}, \tag{62}$$

hold under any of the following conditions:

(a) *if* $p \geqslant 2, \Psi := \Phi^{(p)^*}$ *and* A *or* B^* *is normal,*
(b) *if both* A *and* B^* *are normal,*
(c) *if* $\|\cdot\|_{\Psi} := \|\cdot\|_1$.

Proof: According to the inductive argument, for the proof of (61) it suffices to prove its special case $J := N$ and $M := K + 1$ for $K \leqslant N - 1$. So applying in this case the inequality (3.12) in [11, Th. 3.4] to $C := \mathrm{e}^{-tA}, D := \mathrm{e}^{-tB}$ for an arbitrary $t > 0$, we get

$$\begin{aligned} &\left\| \left(\Gamma^{N-K}_{\mathrm{e}^{-tA^*}, \mathrm{e}^{-tA}}(I) \right)^{1/2} \Gamma^{K}_{\mathrm{e}^{-tA}, \mathrm{e}^{-tB}}(X) \left(\Gamma^{N-K}_{\mathrm{e}^{-tB}, \mathrm{e}^{-tB^*}}(I) \right)^{1/2} \right\|_{\Psi} \\ &\qquad \leqslant \left\| \left(\Gamma^{N-K-1}_{\mathrm{e}^{-tA^*}, \mathrm{e}^{-tA}}(I) \right)^{1/2} \Gamma^{K+1}_{\mathrm{e}^{-tA}, \mathrm{e}^{-tB}}(X) \left(\Gamma^{N-K-1}_{\mathrm{e}^{-tB}, \mathrm{e}^{-tB^*}}(I) \right)^{1/2} \right\|_{\Psi}. \end{aligned} \tag{63}$$

Therefore, the formula (3) in Lemma 2.1, combined with (54) and (53) in Lemma 3.4, justifies (64) and (66) in

$$\left\| \left(\Delta^{N-K}_{A^*,A}(I) \right)^{1/2} \sum_{k=0}^{K} \binom{K}{k} A^{K-k} X B^{k} \left(\Delta^{N-K}_{B,B^*}(I) \right)^{1/2} \right\|_{\Psi}$$

$$= \lim_{t \searrow 0} \left\| \sqrt{t^{K-N} \Gamma^{N-K}_{\mathrm{e}^{-tA^*}, \mathrm{e}^{-tA}}(I)}\, t^{-K} \Gamma^{K}_{\mathrm{e}^{-tA}, \mathrm{e}^{-tB}}(X) \sqrt{t^{K-N} \Gamma^{N-K}_{\mathrm{e}^{-tB}, \mathrm{e}^{-tB^*}}(I)} \right\|_{\Psi} \tag{64}$$

$$\leqslant \lim_{t \searrow 0} \left\| \sqrt{t^{K-N+1} \Gamma^{N-K-1}_{\mathrm{e}^{-tA^*}, \mathrm{e}^{-tA}}(I)}\, t^{-K-1} \Gamma^{K+1}_{\mathrm{e}^{-tA}, \mathrm{e}^{-tB}}(X) \sqrt{t^{K-N+1} \Gamma^{N-K-1}_{\mathrm{e}^{-tB}, \mathrm{e}^{-tB^*}}(I)} \right\|_{\Psi} \tag{65}$$

$$= \left\| \left(\Delta^{N-K-1}_{A^*,A}(I) \right)^{1/2} \sum_{k=0}^{K+1} \binom{K+1}{k} A^{K+1-k} X B^{k} \left(\Delta^{N-K-1}_{B,B^*}(I) \right)^{1/2} \right\|_{\Psi}, \tag{66}$$

while the inequality (65) follows from the inequality (63). ■

Further refinements and enhancements of Theorem 3.6 and [9, Th. 2.15] starts with the following theorem.

Theorem 3.7: *Let* Φ, Ψ *be s.n. functions, let* $A, B, X \in \mathcal{B}(\mathcal{H})$ *be such that* A *is* $(M + I)$*-hyperaccretive,* B^* *is* $(N + J)$*-hyperaccretive for some* $I, J \in \mathbb{N}_{\circ}$ *and* $M, N \in \mathbb{N}$*, such that* $K := \frac{M+N}{2} \in \mathbb{N}$*. If also* $\sum_{k=0}^{K} \binom{K}{k} A^{K-k} X B^{k} \in \mathcal{C}_{\Psi}(\mathcal{H})$*, then for* $c_{M,N} := \frac{\sqrt{(M-1)!(N-1)!}}{(\frac{M+N}{2}-1)!}$

$$\left\| \left(\sum_{m=0}^{M} \binom{M}{m} A^{*m} A^{M-m} \right)^{\frac{1}{2}} \left(X - \sum_{k=0}^{K-1} \frac{T^k}{k!} \mathrm{e}^{-TA} \sum_{i=0}^{k} \binom{k}{i} A^{k-i} X B^{i} \, \mathrm{e}^{-TB} \right) \right.$$

$$\times \left(\sum_{n=0}^{N}\binom{N}{n}B^{N-n}B^{*n}\right)^{\frac{1}{2}}\Bigg\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|(I-\mho^{-}_{A^*,A}(I))^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K--k}XB^{k}(I-\mho^{-}_{B,B^*}(I))^{\frac{1}{2}}\right\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\right\|_{\Psi}, \quad \text{if } I := J := 0; \tag{67}$$

$$\Bigg\|\left(\sum_{m=0}^{M+I}\binom{M+I}{m}A^{*m}A^{M+I-m}\right)^{\frac{1}{2}}\left(X-\sum_{k=0}^{K-1}\frac{T^k}{k!}\,\mathrm{e}^{-TA}\sum_{i=0}^{k}\binom{k}{i}A^{k-i}XB^{i}\mathrm{e}^{-TB}\right)$$

$$\times \left(\sum_{n=0}^{N}\binom{N}{n}B^{N-n}B^{*n}\right)^{\frac{1}{2}}\Bigg\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|\left(\sum_{i=0}^{I}\binom{I}{i}A^{*i}A^{I-i}\right)^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}(I-\mho^{-}_{B,B^*}(I))^{\frac{1}{2}}\right\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|\left(\sum_{i=0}^{I}\binom{I}{i}A^{*i}A^{I-i}\right)^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\right\|_{\Psi}, \quad \text{if } I\in\mathbb{N}, J := 0; \tag{68}$$

$$\Bigg\|\left(\sum_{m=0}^{M}\binom{M}{m}A^{*m}A^{M-m}\right)^{\frac{1}{2}}\left(X-\sum_{k=0}^{K-1}\frac{T^k}{k!}\,\mathrm{e}^{-TA}\sum_{i=0}^{k}\binom{k}{i}A^{k-i}XB^{i}\mathrm{e}^{-TB}\right)$$

$$\times \left(\sum_{n=0}^{N+J}\binom{N+J}{n}B^{N+J-n}B^{*n}\right)^{\frac{1}{2}}\Bigg\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|(I-\mho^{-}_{A^*,A}(I))^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}B^{*j}\right)^{\frac{1}{2}}\right\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}B^{*j}\right)^{\frac{1}{2}}\right\|_{\Psi}, \quad \text{if } I := 0, J\in\mathbb{N}; \tag{69}$$

$$\Bigg\|\left(\sum_{m=0}^{M+I}\binom{M+I}{m}A^{*m}A^{M+I-m}\right)^{\frac{1}{2}}\left(X-\sum_{k=0}^{K-1}\frac{T^k}{k!}\,\mathrm{e}^{-TA}\sum_{i=0}^{k}\binom{k}{i}A^{k-i}XB^{i}\,\mathrm{e}^{-TB}\right)$$

$$\times \left(\sum_{n=0}^{N+J}\binom{N+J}{n}B^{N+J-n}B^{*n}\right)^{\frac{1}{2}}\Bigg\|_{\Psi}$$

$$\leqslant c_{M,N}\left\|\left(\sum_{i=0}^{I}\binom{I}{i}A^{*i}A^{I-i}\right)^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^{k}\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}B^{*j}\right)^{\frac{1}{2}}\right\|_{\Psi}, \quad \text{if } I,J\in\mathbb{N}, \tag{70}$$

hold under any of the following conditions:

(a) *if* $\|\cdot\|_{\Psi} := \|\cdot\|_1$,
(b) *if* $p \geqslant 2$, $\Psi := \Phi^{(p)*}$ *and* A *or* B^* *is normal,*
(c) *if both* A *and* B^* *are normal.*

Proof: The common part of proof for inequalities (67)–(70) relies on the following chain of equalities and inequalities:

$$c_{M,N}^{-1}\left\|\left(\Delta_{A^*,A}^{M+I}(I)\right)^{1/2}\left(X-\sum_{k=0}^{K-1}\frac{T^k}{k!}\,\mathrm{e}^{-TA}\Delta_{A,B}^{k}(X)\,\mathrm{e}^{-TB}\right)\left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}}\right\|_{\Psi} \tag{71}$$

$$=\left\|\int_{[0,T]}\left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}}\frac{t^{\frac{M-1}{2}}}{\sqrt{(M-1)!}}\frac{t^{\frac{N-1}{2}}}{\sqrt{(N-1)!}}\,\mathrm{e}^{-tA}\Delta_{A,B}^{K}(X)\,\mathrm{e}^{-tB}\left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}}\,\mathrm{d}t\right\|_{\Psi} \tag{72}$$

$$\leqslant\left\|\left(\int_{[0,T]}\frac{t^{M-1}}{(M-1)!}\,\mathrm{e}^{-tA^*}\Delta_{A^*,A}^{M+I}(I)\,\mathrm{e}^{-tA}\,\mathrm{d}t\right)^{\frac{1}{2}}\right.$$

$$\left.\times\ \Delta_{A,B}^{K}(X)\left(\int_{[0,T]}\frac{t^{N-1}}{(N-1)!}\,\mathrm{e}^{-tB}\Delta_{B,B^*}^{N+J}(I)\,\mathrm{e}^{-tB^*}\mathrm{d}t\right)^{\frac{1}{2}}\right\|_{\Psi} \tag{73}$$

$$=\left\|\left(\Delta_{A^*,A}^{I}\left(I-\sum_{m=0}^{M-1}\frac{T^m}{m!}\,\mathrm{e}^{-TA^*}\Delta_{A^*,A}^{m}(I)\,\mathrm{e}^{-TA}\right)\right)^{\frac{1}{2}}\Delta_{A,B}^{K}(X)\right.$$

$$\left.\times\left(\Delta_{B,B^*}^{J}\left(I-\sum_{n=0}^{N-1}\frac{T^n}{n!}\,\mathrm{e}^{-TB}\Delta_{B,B^*}^{n}(I)\,\mathrm{e}^{-TB^*}\right)\right)^{\frac{1}{2}}\right\|_{\Psi} \tag{74}$$

$$\leqslant\left\|\left(\Delta_{A^*,A}^{I}\left(I-\mho_{A^*,A}^{-}(I)\right)\right)^{\frac{1}{2}}\Delta_{A,B}^{K}(X)\left(\Delta_{B,B^*}^{J}\left(I-\mho_{B,B^*}^{-}(I)\right)\right)^{\frac{1}{2}}\right\|_{\Psi}$$

$$\leqslant\left\|\left(\Delta_{A^*,A}^{I}(I)\right)^{\frac{1}{2}}\Delta_{A,B}^{K}(X)\left(\Delta_{B,B^*}^{J}(I)\right)^{\frac{1}{2}}\right\|_{\Psi}, \tag{75}$$

where the equality (72) is due to the formula (5) in [9, Lemma 2.4], while the inequality (73) follows by the application of the Cauchy–Schwarz norm inequalities to the s.i. families $A_t := \frac{t^{(M-1)/2}}{\sqrt{(M-1)!}}\left(\Delta_{A^*,A}^{M+I}(I)\right)^{1/2}\mathrm{e}^{-tA}$, $B_t^* := \frac{t^{(N-1)/2}}{\sqrt{(N-1)!}}\left(\Delta_{B,B^*}^{N+J}(I)\right)^{1/2}\mathrm{e}^{-tB^*}$ for $t\in[0,T]$ and to $\Delta_{A,B}^{K}(X)$ instead of X. Namely, in the case (a), we apply the inequality (30) in [5, Th. 3.1(c)], in the case (b), we apply the inequality (32) in [5, Th. 3.1(d)], while in the case (c), we apply the inequality (23) in [3, Th. 3.2]. The equality in (74) is based on formulas (6) and (7) in

[9, Lemma 2.4]. The first inequality in (75) is based on the estimates

$$\Lambda_{A^*,A}^{I,M,T} \overset{\text{def}}{=} \int_{[0,T]} \frac{t^{M-1}}{(M-1)!} \, \mathrm{e}^{-tA^*} \Delta_{A^*,A}^{M+I}(I) \, \mathrm{e}^{-tA} \, \mathrm{d}t = \int_{[0,T]} \frac{t^{M-1}}{(M-1)!} \, \mathrm{e}^{-t\Delta_{A^*,A}} \Delta_{A^*,A}^{M+I} \, \mathrm{d}t(I)$$

$$= \Delta_{A^*,A}^{I} \left(\int_{[0,T]} \frac{t^{M-1}}{(M-1)!} \, \mathrm{e}^{-t\Delta_{A^*,A}} \Delta_{A^*,A}^{M} \, \mathrm{d}t(I) \right) = \Delta_{A^*,A}^{I} \left(I - \sum_{m=0}^{M-1} \frac{T^m}{m!} \, \mathrm{e}^{-TA^*} \Delta_{A^*,A}^{m}(I) \, \mathrm{e}^{-TA} \right) \tag{76}$$

$$= \Delta_{A^*,A}^{I}(I) - \Delta_{A^*,A}^{I} \left(\mathrm{e}^{-TA^*} \mathrm{e}^{-TA} \right) - \sum_{m=1}^{M-1} \frac{T^m}{m!} \, \mathrm{e}^{-TA^*} \Delta_{A^*,A}^{m+I}(I) \, \mathrm{e}^{-TA} \leqslant \Delta_{A^*,A}^{I}(I - \mho_{A^*,A}^{-}(I)), \tag{77}$$

and $\Lambda_{B,B^*}^{J,N,T} \leqslant \Delta_{B,B^*}^{J} \left(I - \mho_{B,B^*}^{-}(I) \right)$ by analogy, where the last equality in (76) derives by the formula (6) in [9, Lemma 2.4], while the inequality (77) is supported by the inequality $\Delta_{A^*,A}^{I} \left(\mathrm{e}^{-TA^*} \mathrm{e}^{-TA} \right) \geqslant \Delta_{A^*,A}^{I}(\mho_{A^*,A}^{-}(I))$ based on (f) in Theorem 2.4 because A is $(I+1)$-hyperaccretive as $I + 1 \leqslant I + M$. Both inequalities in (75) follow by combining those estimates with the (double) monotonicity property (1).

Let us note that $\Delta_{A^*,A}^{I} \left(\mho_{A^*,A}^{-}(I) \right) = I \otimes I \left(\mho_{A^*,A}^{-}(I) \right) = \mho_{A^*,A}^{-}(I)$ for $I := 0$ and $\Delta_{A^*,A}^{I} \left(\mho_{A^*,A}^{-}(I) \right) = 0$ for $I \in \mathbb{N}$ according to (c) in Theorem 2.4, with similar conclusions for $\Delta_{B,B^*}^{J} \left(\mho_{B,B^*}^{-}(I) \right)$, which starting from the already proven inequalities (71)–(75) explains all presented inequalities in (67) if $I := J := 0$, inequalities (68) if $I \in \mathbb{N}, J := 0$, inequalities (69) if $I := 0, J \in \mathbb{N}$ and the inequality (70) if $I, J \in \mathbb{N}$. ∎

Lemma 3.8: *Let Φ, Ψ be s.n. functions, let $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that A is $(M+I)$-hyperaccretive, B^* is $(N+J)$-hyperaccretive for some $M, N, k \in \mathbb{N}$, $I, J \in \mathbb{N}_\circ$, such that $K := \frac{M+N}{2} \in \mathbb{N}$ and $k \leqslant K - 1$. If also $\sum_{k=0}^{K} \binom{K}{k} A^{K-k} X B^k \in \mathfrak{C}_\Psi(\mathcal{H})$, then*

$$\underset{T\to+\infty}{{}^{w}\lim} \left(\Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} \frac{T^k}{k!} \, \mathrm{e}^{-TA} \Delta_{A,B}^{k}(X) \, \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}} = 0, \tag{78}$$

$$\exists V_{A,B}^{M+I,N+J,-} X = \underset{T\to+\infty}{{}^{w}\lim} \left(\Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} \mathrm{e}^{-TA} X \, \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}}, \tag{79}$$

$$\left(\Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} X \left(\Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}} - V_{A,B}^{M+I,N+J,-} X$$
$$= \int_{\mathbb{R}_+} \left(\Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} \frac{t^{K-1}}{(K-1)!} \, \mathrm{e}^{-tA} \Delta_{A,B}^{K}(X) \, \mathrm{e}^{-tB} \left(\Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}} \, \mathrm{d}t, \tag{80}$$

hold under any of the following conditions:

(a) *if $\|\cdot\|_\Psi := \|\cdot\|_1$ and $AX + XB \in \mathfrak{C}_1(\mathcal{H})$,*
(b) *if $p \geqslant 2$, $\Psi := \Phi^{(p)^*}$ and A or B^* is normal,*
(c) *if both A and B^* are normal.*

Moreover, $V_{A,B}^{M+I,N+J,-} X = 0$ in the cases (b) and (c).

Proof: To prove formula (78), let us first consider the case $2 \leqslant M \leqslant N$. If $k \leqslant M - 1$ then we take $m := n := k$, while in the case $M \leqslant k \leqslant \frac{M+N}{2} - 1$ let us take $m := M - 1$ and

$n := 2k - M + 1$ implying that $3 \leqslant 2M - M + 1 \leqslant 2k - M + 1 \leqslant M + N - 2 - M + 1 = N - 1$. In both cases we have $m \leqslant M - 1$, $n \leqslant N - 1$ and $m + n = 2k$, so

$$\begin{aligned} &{}^{s}\lim_{T\to+\infty} \big(\Delta^{M+I}_{A^*,A}(I)\big)^{\frac12} \frac{T^k}{k!} \,\mathrm{e}^{-TA}\Delta^k_{A,B}(X)\,\mathrm{e}^{-TB} \big(\Delta^{N+J}_{B,B^*}(I)\big)^{\frac12} \\ &= \frac{\sqrt{m!n!}}{k!} \;{}^{s}\lim_{T\to+\infty} \left(\frac{T^m}{m!}\Delta^{M+I}_{A^*,A}(I)\right)^{\frac12} \mathrm{e}^{-TA}\Delta^k_{A,B}(X)\,\mathrm{e}^{-TB} \left(\frac{T^n}{n!}\Delta^{N+J}_{B,B^*}(I)\right)^{\frac12} = 0. \end{aligned} \tag{81}$$

Namely, ${}^{s}\lim_{T\to+\infty} \left(\frac{T^m}{m!}\Delta^{M+I}_{A^*,A}(I)\right)^{\frac12} \mathrm{e}^{-TA} = 0$, according to the formula (9) in Theorem 2.4 as $m \leqslant M - 1 < M + I$. Similarly, ${}^{s}\lim_{T\to+\infty} \left(\frac{T^n}{n!}\Delta^{N+J}_{B,B^*}(I)\right)^{\frac12} \mathrm{e}^{-TB^*} = 0$ implying that ${}^{s}\lim_{T\to+\infty} \mathrm{e}^{-TB} \left(\frac{T^n}{n!}\Delta^{N+J}_{B,B^*}(I)\right)^{\frac12} = 0$ if B is normal, in which case the last equality in (81) is valid, thus supporting the formula (78) in the case (c), as well as in the case (b) if B is normal. If A is normal in the remaining case in (b), then applying Lemma 2.1(d) to $A_T := \left(\frac{T^m}{m!}\Delta^{M+I}_{A^*,A}(I)\right)^{\frac12} \mathrm{e}^{-TA}$ and $B_T := \Delta^k_{A,B}(X)\,\mathrm{e}^{-TB} \left(\frac{T^n}{n!}\Delta^{N+J}_{B,B^*}(I)\right)^{\frac12}$ for all $T \geqslant 0$ implies the formula (78).

In the next case $M := 1$ we have $N \geqslant 3$, so $2k \leqslant 2K - 2 = N - 1$ and

$$\begin{aligned} &{}^{w}\lim_{T\to+\infty} \big(\Delta^{I+1}_{A^*,A}(I)\big)^{1/2} \frac{T^k}{k!} \,\mathrm{e}^{-TA}\Delta^k_{A,B}(X)\,\mathrm{e}^{-TB} \big(\Delta^{N+J}_{B,B^*}(I)\big)^{1/2} \\ &= \frac{\sqrt{(2k)!}}{k!} \;{}^{w}\lim_{T\to+\infty} \big(\Delta^{I+1}_{A^*,A}(I)\big)^{1/2} \mathrm{e}^{-TA}\Delta^k_{A,B}(X)\,\mathrm{e}^{-TB} \left(\frac{T^{2k}}{(2k)!}\Delta^{N+J}_{B,B^*}(I)\right)^{1/2} = 0, \end{aligned} \tag{82}$$

if A or B is normal. Indeed, if B is normal, then (9) applied to B^* instead of A implies ${}^{s}\lim_{T\to+\infty} \mathrm{e}^{-TB} \left(\frac{T^{2k}}{(2k)!}\Delta^{N+J}_{B,B^*}(I)\right)^{1/2} = {}^{s}\lim_{T\to+\infty} \left(\frac{T^{2k}}{(2k)!}\Delta^{N+J}_{B,B^*}(I)\right)^{1/2} \mathrm{e}^{-TB^*} = 0$, so the equality in (82) follows as all others operators appearing in (82) are uniformly bounded for $T \in \mathbb{R}_+$. Also, if A is normal, then for any $h \in \mathcal{H}$, the spectral measure E associated to $A^* + A$ and a finite measure μ_h given by $\mu_h(\delta) \stackrel{\text{def}}{=} \langle E(\delta)h, h\rangle$ for any Borel set $\delta \subset \mathbb{R}_+$ we have$\lim_{T\to+\infty} \left\| \big(\Delta^{I+1}_{A^*,A}(I)\big)^{\frac12} \mathrm{e}^{-TA} h \right\|^2 = \lim_{T\to+\infty} \left\| (A^* + A)^{\frac{I+1}{2}} \mathrm{e}^{-T\frac{A^*+A}{2}} h \right\|^2 = \lim_{T\to+\infty} \int_0^{\|A^*+A\|} t^{I+1}\,\mathrm{e}^{-Tt}\,\mathrm{d}\mu_h(t) = \int_0^{\|A^*+A\|} 0\mu_h(t) = 0$ according to the Lebesgue dominance theorem. Thus ${}^{s}\lim_{T\to+\infty}\big(\Delta^{I+1}_{A^*,A}(I)\big)^{\frac12} \mathrm{e}^{-TA} = 0$, which combined by the fact that ${}^{w}\lim_{T\to+\infty} \mathrm{e}^{-TB} \left(\frac{T^{2k}}{(2k)!}\Delta^{N+J}_{B,B^*}(I)\right)^{1/2} = 0$, based on (9) applied to B^* instead of A, which proves (82), according to Lemma 2.1(d).

In the case (a) note that for rank one operators $g^\star \otimes h\colon \mathcal{H} \to \mathcal{H}\colon f \mapsto \langle f, g\rangle h$, where $g, h \in \mathcal{H}$, we have

$$\begin{aligned} &\lim_{T\to+\infty} \left\| \big(\Delta^{M+I}_{A^*,A}(I)\big)^{1/2} \frac{T^k}{k!} \,\mathrm{e}^{-TA}(g^\star \otimes h)\,\mathrm{e}^{-TB} \big(\Delta^{N+J}_{B,B^*}(I)\big)^{1/2} \right\|_1 \\ &= \frac{\sqrt{m!n!}}{k!} \lim_{T\to+\infty} \left\| \left(\left(\frac{T^n}{n!}\Delta^{N+J}_{B,B^*}(I)\right)^{1/2} \mathrm{e}^{-TB^*} g \right)^\star \otimes \left(\left(\frac{T^m}{m!}\Delta^{M+I}_{A^*,A}(I)\right)^{1/2} \mathrm{e}^{-TA} h \right) \right\|_1 \end{aligned}$$

$$= \frac{\sqrt{m!n!}}{k!} \lim_{T\to+\infty} \left\| \left(\frac{T^n}{n!} \Delta_{B,B^*}^{N+J}(I) \right)^{1/2} \mathrm{e}^{-TB^*} g \right\| \left\| \left(\frac{T^m}{m!} \Delta_{A^*,A}^{M+I}(I) \right)^{1/2} \mathrm{e}^{-TA} h \right\| = 0, \quad (83)$$

where the last equality in (83) is based on the formula (9) in Theorem 2.4. Thus formula (83) actually shows that

$$\mathfrak{e}_{1(\mathcal{H})}\lim_{T\to+\infty} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{1/2} \frac{T^k}{k!} \mathrm{e}^{-TA} Y \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{1/2} = 0 \quad (84)$$

for all finite rank operators Y on $\mathcal{H}$. As $\lim_{T\to+\infty} \left\| \left(\frac{T^m}{m!} \Delta_{A^*,A}^{M+I}(I) \right)^{1/2} \mathrm{e}^{-TA} h \right\| = 0$ and functions $T \mapsto \left\| \left(\frac{T^m}{m!} \Delta_{A^*,A}^{M+I}(I) \right)^{1/2} \mathrm{e}^{-TA} h \right\|$ are continuous on $\mathbb{R}_+$ for any $h \in \mathcal{H}$, then $\sup_{T\geqslant 0} \left\| \left(\frac{T^m}{m!} \Delta_{A^*,A}^{M+I}(I) \right)^{1/2} \mathrm{e}^{-TA} h \right\| < +\infty$, and thus $\sup_{T\geqslant 0} \left\| \left(\frac{T^m}{m!} \Delta_{A^*,A}^{M+I}(I) \right)^{1/2} \mathrm{e}^{-TA} \right\| < +\infty$ by the uniform boundedness principle. Similarly, $\sup_{T\geqslant 0} \left\| \left(\frac{T^n}{n!} \Delta_{B,B^*}^{N+J}(I) \right)^{1/2} \mathrm{e}^{-TB^*} \right\| < +\infty$, so it follows that the transformer family $\left\{ \frac{T^k}{k!} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{1/2} \mathrm{e}^{-TA} \otimes \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} \right\}_{T>0}$ is uniformly bounded on $\mathfrak{C}_1(\mathcal{H})$. Thus, by a convergence principle of Banach-Steinhaus theorem it follows that formula (84) remains valid for all $Y \in \mathfrak{C}_1(\mathcal{H})$, including $Y := \Delta_{A,B}^k(X)$, which completes the proof for the formula (78) if $2 \leqslant M \leqslant N$ or $M := 1, N \geqslant 3$.

The remaining case $M > N$ proves by analogy.

To prove (79), let us first note that

$$\begin{aligned}
&{}^{w}\lim_{T\to+\infty} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} \mathrm{e}^{-TA} X \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} \\
&= {}^{w}\lim_{T\to+\infty} \sum_{k=0}^{K-1} \frac{T^k}{k!} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} \mathrm{e}^{-TA} \Delta_{A,B}^k(X) \mathrm{e}^{-TB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} && (85) \\
&= {}^{w}\lim_{T\to+\infty} \Bigg(\left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} X \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} - \\
&\quad \int_{[0,T]} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} \frac{t^{K-1}}{(K-1)!} \mathrm{e}^{-tA} \Delta_{A,B}^K(X) \mathrm{e}^{-tB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} \,\mathrm{d}t \Bigg) && (86) \\
&= \left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} X \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} - \\
&\quad \int_{\mathbb{R}_+} \left(\Delta_{A^*,A}^{M+I}(I)\right)^{\frac{1}{2}} \frac{t^{K-1}}{(K-1)!} \mathrm{e}^{-tA} \Delta_{A,B}^K(X) \mathrm{e}^{-tB} \left(\Delta_{B,B^*}^{N+J}(I)\right)^{\frac{1}{2}} \,\mathrm{d}t, && (87)
\end{aligned}$$

where the equality in (85) follows from the formula (78), the equality in (86) is based on the formula (5) in [9, Lemma 2.4], while the equality (87) is due to [5, Th. 3.1(c)] in the case (a), [5, Th. 3.1(d)] in the case (b), [5, Lemma 2.1(b)] in the case (c), all applied to s.i. families $A_t := \left(\frac{t^{M-1}}{(M-1)!} \Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} \mathrm{e}^{-tA}$ and $B_t^* := \left(\frac{t^{N-1}}{(N-1)!} \Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}} \mathrm{e}^{-tB^*}$ for all $t \in \mathbb{R}_+$ and to

$\Delta^K_{A,B}(X)$ instead of X therein. So the equalities (85)–(87) prove the integral representation formula (80).

To prove the last statement in the theorem, note that if B is normal, then it follows from the case $\alpha := \frac{1}{2}$ in [9, Lemma 2.5(d)] that ${}^s\lim_{T\to+\infty} \sqrt{B+B^*}\,\mathrm{e}^{-TB^*} = 0$, and therefore ${}^s\lim_{T\to+\infty} \mathrm{e}^{-TB}\left(\Delta^{N+J}_{B,B^*}(I)\right)^{\frac{1}{2}} = {}^s\lim_{T\to+\infty} \mathrm{e}^{-TB}(B+B^*)^{\frac{N+J}{2}} = 0$. As $\left\|\left(\Delta^{M+I}_{A^*,A}(I)\right)^{\frac{1}{2}} \mathrm{e}^{-tA}X\right\| \leqslant (2\|A\|)^{M+I}\|X\| < +\infty$, this implies $V^{M+I,N+J,-}_{A,B}X = 0$. Similarly, if A is normal, then for all $g,h \in \mathcal{H}$ we have $\left|\left\langle\left(\Delta^{M+I}_{A^*,A}(I)\right)^{\frac{1}{2}} \mathrm{e}^{-TA}X\,\mathrm{e}^{-TB}\left(\Delta^{N+J}_{B,B^*}(I)\right)^{\frac{1}{2}} g, h\right\rangle\right| \leqslant (2\|B\|)^{N+J}\|X\|\|g\|\left\|\mathrm{e}^{-TA^*}(A^*+A)^{\frac{M+I}{2}}h\right\| \to 0$ as $T \to +\infty$, so again $V^{M+I,N+J,-}_{A,B}X = 0$. ∎

Theorem 3.7 and Lemma 3.8 provides new tools for further advanced versions of Young and A-G norm inequalities for N-hyperaccretive operators, already obtained in Theorem 3.6, [9, Th. 2.15] and [12, Th. 3.6], as we show in the following theorem.

Theorem 3.9: *Under the conditions of Theorem 3.7, including conditions (b), (c) therein, and the condition (a) of Lemma 3.8*

$$\begin{aligned}
&\left\|\left(\sum_{m=0}^{M+I}\binom{M+I}{m}A^{*m}A^{M+I-m}\right)^{\frac{1}{2}} X\left(\sum_{n=0}^{N+J}\binom{N+J}{n}B^{N+J-n}B^{*n}\right)^{\frac{1}{2}} - V^{M+I,N+J,-}_{A,B}X\right\|_\Psi\\
&\leqslant c_{M,N}\left\|\left(\sum_{i=0}^{I}\binom{I}{i}A^{*i}\left(I-\mho^-_{A^*,A}(I)\right)A^{I-i}\right)^{\frac{1}{2}}\right.\\
&\quad\left.\times\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^k\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}\left(I-\mho^-_{B,B^*}(I)\right)B^{*j}\right)^{\frac{1}{2}}\right\|_\Psi\\
&\leqslant c_{M,N}\left\|\left(\sum_{i=0}^{I}\binom{I}{i}A^{*i}A^{I-i}\right)^{\frac{1}{2}}\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^k\left(\sum_{j=0}^{J}\binom{J}{j}B^{J-j}B^{*j}\right)^{\frac{1}{2}}\right\|_\Psi. \qquad (88)
\end{aligned}$$

Moreover, $V^{M+I,N+J,-}_{A,B}X = 0$ *in the cases (b) and (c).*

Proof: The proof of the first inequality in (88) relies on the following chain of equalities and inequalities:

$$\begin{aligned}
&c^{-1}_{M,N}\left\|\left(\Delta^{M+I}_{A^*,A}(I)\right)^{\frac{1}{2}} X\left(\Delta^{N+J}_{B,B^*}(I)\right)^{\frac{1}{2}} - V^{M+I,N+J,-}_{A,B}X\right\|_\Psi\\
&= \left\|\int_{\mathbb{R}_+}\left(\Delta^{M+I}_{A^*,A}(I)\right)^{\frac{1}{2}}\frac{t^{\frac{M-1}{2}}}{\sqrt{(M-1)!}}\frac{t^{\frac{N-1}{2}}}{\sqrt{(N-1)!}}\,\mathrm{e}^{-tA}\Delta^K_{A,B}(X)\,\mathrm{e}^{-tB}\left(\Delta^{N+J}_{B,B^*}(I)\right)^{\frac{1}{2}}\mathrm{d}t\right\|_\Psi \qquad (89)
\end{aligned}$$

$$\leqslant \left\| \left(\int_{\mathbb{R}_+} \frac{t^{M-1}}{(M-1)!} \, e^{-tA^*} \Delta_{A^*,A}^{M+I}(I) \, e^{-tA} \, dt \right)^{\frac{1}{2}} \Delta_{A,B}^{K}(X) \right.$$

$$\left. \times \left(\int_{\mathbb{R}_+} \frac{t^{N-1}}{(N-1)!} \, e^{-tB} \Delta_{B,B^*}^{N+J}(I) \, e^{-tB^*} dt \right)^{\frac{1}{2}} \right\|_{\Psi} \tag{90}$$

$$= \left\| \left(\Delta_{A^*,A}^{I} \left(I - \mho_{A^*,A}^{-}(I) \right) \right)^{\frac{1}{2}} \Delta_{A,B}^{K}(X) \left(\Delta_{B,B^*}^{J} \left(I - \mho_{B,B^*}^{-}(I) \right) \right)^{\frac{1}{2}} \right\|_{\Psi}, \tag{91}$$

where the equality (89) is based on the integral representation formula (80) in Lemma 3.8, while the inequality (90) follows by the application of the Cauchy–Schwarz norm inequalities to s.i. families $A_t := \frac{t^{(M-1)/2}}{\sqrt{(M-1)!}} \left(\Delta_{A^*,A}^{M+I}(I) \right)^{\frac{1}{2}} e^{-tA}$, $B_t^* := \frac{t^{(N-1)/2}}{\sqrt{(N-1)!}} \left(\Delta_{B,B^*}^{N+J}(I) \right)^{\frac{1}{2}} e^{-tB^*}$ for $t \in \mathbb{R}_+$ and to $\Delta_{A,B}^{K}(X)$ instead of X. Namely, in the case (a) we apply the inequality (30) in [5, Th. 3.1(c)], in the case (b), we apply the inequality (32) in [5, Th. 3.1(d)], while in the case (c) we apply the inequality (23) in [3, Th. 3.2]. The equality in (91) is based on the formula (7) in Theorem 2.4, as $\int_{\mathbb{R}_+} \frac{t^{M-1}}{(M-1)!} \, e^{-tA^*} \Delta_{A^*,A}^{M+I}(I) \, e^{-tA} \, dt = \Delta_{A^*,A}^{I} \left(\int_{\mathbb{R}_+} \frac{t^{M-1}}{(M-1)!} \, e^{-tA^*} \Delta_{A^*,A}^{M}(I) \, e^{-tA} \, dt \right) = \Delta_{A^*,A}^{I} \left(I - \mho_{A^*,A}^{-}(I) \right)$, and similarly for B.

The last inequality in (88) is in fact the already proven last inequality in (75). The last statement of this theorem is exactly the last statement of Lemma 3.8. ■

The extension of Theorem 3.6 is another direction is provided by the following theorem.

Theorem 3.10: *Let $K \in \mathbb{N}_\circ, M, N \in \mathbb{N}$ satisfy $0 \leqslant K \leqslant M \leqslant N$, let $c_\circ > 0$ and $\{c_n\}_{n=1}^{N}$ be a sequence in $\mathbb{R}_+$. Let also P_K and T_{N-K} be polynomials given by $P_K\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto c_\circ \prod_{k=1}^{K} (c_k + z)$ and $T_{N-K}\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto \prod_{k=K+1}^{N} (c_k + z)$ and let Φ, Ψ be a s.n. functions. If $A, B, X \in \mathfrak{B}(\mathcal{H})$ are such that A, B^* are N-hyperaccretive, $X \in \mathfrak{C}_\Psi(\mathcal{H})$, then*

$$\left\| \sqrt{T_{N-K}(A^* \otimes I + I \otimes A)(I)} P_K(A \otimes I + I \otimes B)(X) \sqrt{T_{N-K}(B \otimes I + I \otimes B^*)(I)} \right\|_{\Psi}$$
$$\leqslant \left\| \sqrt{T_{N-M}(A^* \otimes I + I \otimes A)(I)} P_M(A \otimes I + I \otimes B)(X) \sqrt{T_{N-M}(B \otimes I + I \otimes B^*)(I)} \right\|_{\Psi}, \tag{92}$$

$$\left\| \sqrt{P_N(A^* \otimes I + I \otimes A)(I)} X \sqrt{P_N(B \otimes I + I \otimes B^*)(I)} \right\|_{\Psi} \leqslant \| P_N(A \otimes I + I \otimes B)(X) \|_{\Psi}, \tag{93}$$

hold under any of the following conditions:

(a) *if $\|\cdot\|_\Psi = \|\cdot\|_1$,*
(b) *if $p \geqslant 2, \Psi := \Phi^{(p)^*}$ and A or B^* is normal,*
(c) *if both A and B^* are normal operators.*

Proof: As it suffices to prove the case $c_\circ := 1$, the application of the inequality (3.11) in [11, Th. 3.4] to $C := \mathrm{e}^{-tA}$, $D := \mathrm{e}^{-tB}$ and $r_n := \mathrm{e}^{-tc_n}$ for all $\mathbb{N} \ni n \leqslant N$ and $t > 0$ provides

$$\left\| \left(\prod_{k=K+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA^*} \otimes \mathrm{e}^{-tA})(I) \right)^{\frac{1}{2}} \prod_{k=1}^{K} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA} \otimes \mathrm{e}^{-tB})(X) \right.$$

$$\left. \times \left(\prod_{k=K+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tB} \otimes \mathrm{e}^{-tB^*})(I) \right)^{\frac{1}{2}} \right\|_{\Psi}$$

$$\leqslant \left\| \left(\prod_{k=M+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA^*} \otimes \mathrm{e}^{-tA})(I) \right)^{\frac{1}{2}} \prod_{k=1}^{M} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA} \otimes \mathrm{e}^{-tB})(X) \right.$$

$$\left. \times \left(\prod_{k=M+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tB} \otimes \mathrm{e}^{-tB^*})(I) \right)^{\frac{1}{2}} \right\|_{\Psi}. \tag{94}$$

Next, we introduce families $\mathcal{T}_n(t) \stackrel{\text{def}}{=} \prod_{k=1}^{n} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA} \otimes \mathrm{e}^{-tB})(X)$, $\mathcal{Q}_n(t) \stackrel{\text{def}}{=} \prod_{k=n+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tA^*} \otimes \mathrm{e}^{-tA})(I)$, $\mathcal{S}_n(t) \stackrel{\text{def}}{=} \prod_{k=n+1}^{N} t^{-1}(I \otimes I - \mathrm{e}^{-tc_k}\, \mathrm{e}^{-tB} \otimes \mathrm{e}^{-tB^*})(I)$ for all $\mathbb{N} \ni n \leqslant N$ and $t > 0$, so the formula (60) in Lemma 3.5 actually says that ${}^{\Psi}\lim_{t \searrow 0} \mathcal{T}_n(t) = P_n(\Delta_{A,B})(X)$, ${}^{\mathcal{B}(\mathcal{H})}\lim_{t \searrow 0} \mathcal{Q}_n(t) = T_{N-n}(\Delta_{A^*,A})(I)$ and ${}^{\mathcal{B}(\mathcal{H})}\lim_{t \searrow 0} \mathcal{S}_n(t) = T_{N-n}(\Delta_{B,B^*})(I)$. Therefore by Ando [13, Th. 1], we get ${}^{\mathcal{B}(\mathcal{H})}\lim_{t \searrow 0} \sqrt{\mathcal{Q}_n(t)} = \sqrt{T_{N-n}(\Delta_{A^*,A})(I)}$ and ${}^{\mathcal{B}(\mathcal{H})}\lim_{t \searrow 0} \sqrt{\mathcal{S}_n(t)} = \sqrt{T_{N-n}(\Delta_{B,B^*})(I)}$, implying that for $n \in \{K, M\}$

$${}^{\Psi}\lim_{t \searrow 0} \sqrt{\mathcal{Q}_n(t)}\mathcal{T}_n(t)\sqrt{\mathcal{S}_n(t)} = \sqrt{T_{N-n}(\Delta_{A^*,A})(I)}P_n(\Delta_{A,B})(X)\sqrt{T_{N-n}(\Delta_{B,B^*})(I)}, \tag{95}$$

$$\lim_{t \searrow 0} \left\| \sqrt{\mathcal{Q}_n(t)}\mathcal{T}_n(t)\sqrt{\mathcal{S}_n(t)} \right\|_{\Psi} = \left\| \sqrt{T_{N-n}(\Delta_{A^*,A})(I)}P_n(\Delta_{A,B})(X)\sqrt{T_{N-n}(\Delta_{B,B^*})(I)} \right\|_{\Psi}. \tag{96}$$

Indeed, the application of the formula (3) in Lemma 2.1 to $A_t := \sqrt{\mathcal{Q}_n(t)}$, $B_t := \sqrt{\mathcal{S}_n(t)}$ $X_t := \mathcal{T}_n(t)$ implies the equality (95), and therefore the continuity of $\|\cdot\|_\Psi$ norm on $\mathfrak{C}_\Psi(\mathcal{H})$ proves the formula (96). Thus the limiting process based on the formula (96), applied to the inequality (94), proves the inequality (92).

Special case $K := 0$ and $M := N$ (in which case we assume $P_\circ \equiv c_\circ$) of the inequality (92) asserts the inequality (93). ■

We conclude this paper with the following.

Open question: Does the Theorem 3.7 remains valid if the requirement for normality therein is reduced to quasinormality for $N \geqslant 2$?

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