J. OPERATOR THEORY
31(1994), 3–10

SOME PERTURBATION INEQUALITIES FOR SELF-ADJOINT OPERATORS

DANKO JOCIĆ and FUAD KITTANEH

1. INTRODUCTION

In a recent paper, extending the results of several mathematicians and physicists, Ando [1] proved, among other things, that if A, B are positive semi-definite compact operators and if $|||\cdot|||$ is any unitarily invariant norm, then

$$||| \, |A-B|^p \, ||| \leqslant |||A^p - B^p||| \quad \text{for all } p \geqslant 1,$$

where $|X|$ denote the modulus $(X^*X)^{1/2}$ (see also [5] and [11]).

In this note we are concerned with the comparison of $|||(A-B)^{2n+1}|||$ and $|||A^{2n+1} - B^{2n+1}|||$ for an arbitrary pair of self-adjoint operators and any integer $n \geqslant 1$. Our (best possible) estimate asserts that $|||(A-B)^{2n+1}|||$ is dominated by $2^{2n}|||A^{2n+1} - B^{2n+1}|||$. In particular, if $A^{2n+1} - B^{2n+1}$ belongs to the norm ideal $\mathcal{J}_{|||\cdot|||}$ associated with the norm $|||\cdot|||$, then $(A-B)^{2n+1}$ belongs to $\mathcal{J}_{|||\cdot|||}$. Specializing this perturbation result to the particularly important ideals $\mathcal{J}_p$ (the Schatten p-classes for $p \geqslant 1$), one can easily see that $A - B$ belongs to $\mathcal{J}_{(2n+1)p}$ whenever $A^{2n+1} - B^{2n+1}$ belongs to $\mathcal{J}_p$. This is a considerable strengthening of an earlier result of Koplienko [9], which shows that if $A^{2n+1} - B^{2n+1}$ belongs to $\mathcal{J}_p$, then $A - B$ belongs to $\mathcal{J}_q$ for all $q > 2(2n+1)p$.

2. PRELIMINARIES.

Let $B(H)$ denote the space of all bounded linear operators on a Hilbert space H. Besides the usual operator norm $\|\cdot\|$, there are other interesting norms defined on ideals contained in the ideal of compact operators. For any compact operator A, let $s_1(A) \geqslant s_2(A) \geqslant \cdots$ denote the singular values of A (i.e. the eigenvalues of $|A|$), arranged in non-increasing order with multiplicities counted. Each "symmetric gauge function" Φ on sequences, gives rise to a symmetric norm or a unitarily invariant norm on operators defined by $\|A\|_\Phi = \Phi(\{s_j(A)\})$. We will denote the symbol $|||\cdot|||$ any such norm. Each such norm is defined on a natural subclass $\mathcal{J}_{|||\cdot|||}$ of $B(H)$ called the norm ideal associated with the norm $|||\cdot|||$ and satisfies the invariance property $|||UAV||| = |||A|||$ for all A in this ideal and for all unitary operators U, V. Each norm ideal $\mathcal{J}_{|||\cdot|||}$ is closed in the topology generated by the norm $|||\cdot|||$.

Specially well known among these norms are Schatten p-norms defined as $\|A\|_p = \left(\sum_j s_j(A)^p\right)^{1/p}$ for $1 \leqslant p \leqslant \infty$. The associated ideals called the *Schatten p-classes*, are denoted by $\mathcal{J}_p$, $1 \leqslant p \leqslant \infty$, where by convention $\mathcal{J}_\infty$ is the ideal of compact operators and $\|A\|_\infty = \max s_j(A) = s_1(A) = \|A\|$.

The Ky Fan norms defined as $\|A\|_k = \sum_{j=1}^{k} s_j(A)$, $k = 1, 2, \ldots$, represent another interesting family of unitarily invariant norms. The importance of this family lies in the Ky Fan dominance property which says that if Y belongs to $\mathcal{J}_{|||\cdot|||}$ and if X is a compact operator such that $\|X\|_k \leqslant \|Y\|_k$ for $k = 1, 2, \ldots$, then X also belongs to $\mathcal{J}_{|||\cdot|||}$ and $|||X||| \leqslant |||Y|||$. For a complete account of the theory of norm ideals, the reader is referred to [6], [13] or [14].

The following operator form of the arithmetic-geometric mean inequality plays a central role in our analysis. For arbitrary operators A, B, X, and for every unitarily invariant norm we have

$$2|||A^*XB||| \leqslant |||AA^*X + XBB^*|||. \tag{1}$$

See [2], [4], [7] and [10] for comprehensive discussions of (1) and related inequalities.

3. MAIN RESULTS.

We begin with a lemma of independent interest.

LEMMA. *Let A, B and X be operators in $B(H)$ such that A and B are self--adjoint. Then for every integer $n \geqslant 1$ and for every unitarily invariant norm we have the following chain of inequalities*

$$\begin{aligned} |||A^n(AX - XB)B^n||| &\leqslant |||A^{n-1}(A^3X - XB^3)B^{n-1}||| \leqslant \cdots \leqslant \\ &\leqslant |||A(A^{2n-1}X - XB^{2n-1})B||| \leqslant |||A^{2n+1}X - XB^{2n+1}|||. \end{aligned} \tag{2}$$

Proof. First we establish the desired inequalities for the usual operator norm. Let $\mathbf{Z}_{2n+1} = \{0, 1, 2, \ldots, 2n\}$ be the complete set of residues modulo $2n+1$ and for every integer m let $m \bmod (2n+1)$ stand for the (unique) residue m_0 in $\mathbf{Z}_{2n+1}$ such that $2n+1$ divides $m - m_0$. Also let $k_i = \min\{2^i \bmod (2n+1), -2^i \bmod (2n+1)\}$ for $i = 0, 1, 2, \ldots$. Note that $1 \leqslant k_i \leqslant n$. We will prove that $k_{i_0} = 1$ for some $i_0 > 0$. Since $\{k_i : i \geqslant 0\}$ is finite, it follows that $k_j = k_{j'}$, for some indices j and j' with $j < j'$. Thus, we have $2^j \bmod (2n+1) = 2^{j'} \bmod (2n+1)$ or $2^j \bmod (2n+1) = -2^{j'} \bmod (2n+1)$, and so $2n+1$ divides either $2^j(2^{j'-j} - 1)$ or $2^j(2^{j'-j} + 1)$. Hence $2n+1$ divides either $2^{j'-j} - 1$ or $2^{j'-j} + 1$, and so if we set $i_0 = j' - j$, then we have $k_{i_0} = k_0 = 1$.

Let $c_i = \|A^{k_i}(A^{2n-2k_i+1}X - XB^{2n-2k_i+1})B^{k_i}\|$ for $i \geqslant 0$, and $c = \|A^{2n+1}X - XB^{2n+1}\|$. Then utilizing the inequality (1) we get

$$\begin{aligned} c_i &= \|A^{k_i}(A^{2n-2k_i+1}X - XB^{2n-2k_i+1})B^{k_i}\| \leqslant \\ &\leqslant \frac{1}{2}\|A^{2k_i}(A^{2n-2k_i+1}X - XB^{2n-2k_i+1}) + (A^{2n-2k_i+1}X - XB^{2n-2k_i+1})B^{2k_i}\| = \\ &= \|\frac{1}{2}(A^{2n+1}X - XB^{2n+1}) + \\ &\quad \operatorname{sign}(2n - 4k_i + 1)A^{k_{i+1}}(A^{2n-2k_{i+1}+1}X - XB^{2n-2k_{i+1}+1})B^{k_{i+1}}\| \leqslant \\ &\leqslant \frac{c}{2} + \frac{c_{i+1}}{2}. \end{aligned} \tag{3}$$

It should be noted here that $k_{i+1} = \begin{cases} 2k_i & \text{if } k_i \leqslant \frac{n}{2} \\ 2n - 2k_i + 1 & \text{if } k_i > \frac{n}{2}. \end{cases}$

By induction, we obtain from (3) that

$$c_0 \leqslant (1 - 2^{-i})c + 2^{-i}c_i \quad \text{for all } i \geqslant 1. \tag{4}$$

Since $k_{i_0} = k_0 = 1$, it follows that $c_{i_0} = c_0$ and so, by (4) applied to i_0, we obtain $c_0 \leqslant (1 - 2^{-i_0})c + 2^{-i_0}c_0$.

Therefore $c_0 \leqslant c$, and so we have

$$\|A(A^{2n-1}X - XB^{2n-1})B\| \leqslant \|A^{2n+1}X - XB^{2n+1}\|,$$

which is the rightmost inequality in (2). Now replacing X by A^kXB^k and n by $n-k$ in this inequality enables us to obtain the other inequalities in (2) for the usual operator norm case.

Next, we go from this to the case of any other unitarily invariant norm by a familiar procedure. Assume that $A^{2n+1}X - XB^{2n+1}$ belongs to $\mathcal{J}_{|||\cdot|||}$ (otherwise the assertion of the lemma is trivially satisfied).

Then in particular $A^{2n+1}X - XB^{2n+1}$ is compact and so $\pi(A)^{2n+1}\pi(X) - \pi(X)\pi(B)^{2n+1} = 0$, where $\pi : B(H) \to B(H)/\mathcal{J}_\infty$ is the canonical projection of $B(H)$ onto the Calkin algebra $B(H)/\mathcal{J}_\infty$, which is a C^*-algebra and hence it can be represented as an operator algebra. Consequently, by the usual operator norm case of (2), it follows that

$$\pi(A)^k(\pi(A)^{2n-2k+1}\pi(X) - \pi(X)\pi(B)^{2n-2k+1})\pi(B)^k = 0 \quad \text{for all } 1 \leqslant k \leqslant n,$$

and hence $A^k(A^{2n-2k+1}X - XB^{2n-2k+1})B^k$ is compact for all $1 \leqslant k \leqslant n$.

But now, by a similar argument as in the case of the usual operator norm, one can establish the inequalities in (2) for the Ky Fan norms $\|\cdot\|_k$ for $k = 1, 2, \ldots$ and invoke the Ky Fan dominance property to conclude (2) for all unitarily invariant norms. ∎

Now we are in a position to prove our main result.

THEOREM. *Let A, B be self-adjoint operators in $B(H)$. Then for every integer $n \geqslant 1$ and for every unitarily invariant norm we have*

$$|||(A-B)^{2n+1}||| \leqslant 2^{2n}|||A^{2n+1} - B^{2n+1}|||. \tag{5}$$

Proof. First we prove (5) for the usual operator norm. According to the lemma above with $X = I$ (the identity operator), we have

$$2\|A^{2n+1} - B^{2n+1}\| \geqslant \|A^{2n+1} - B^{2n+1}\| + \|A(A^{2n-1} - B^{2n-1})B\| \geqslant$$
$$\geqslant \|A^{2n}(A-B) + (A-B)B^{2n}\|.$$

Since $A - B$ is self-adjoint, it follows that there exists a sequence $\{x_i\}$ of unit vectors in H such that $(A-B)x_i - \lambda x_i \to 0$ as $i \to \infty$, where $|\lambda| = \|A - B\|$. Consequently, we have

$$\|A^{2n}(A-B) + (A-B)B^{2n}\| \geqslant |(A^{2n}(A-B)x_i, x_i) + ((A-B)B^{2n}x_i, x_i)| \geqslant$$
$$\geqslant |\lambda|(\|A^n x_i\|^2 + \|B^n x_i\|^2) - (\|A^{2n}\| + \|B^{2n}\|)\|(A-B)x_i - \lambda x_i\|.$$

But for every self-adjoint operator T and for every unit vector x in H, we have

$$\|T^n x\|^2 \geqslant \|Tx\|^{2n} \quad \text{for } n \geqslant 1. \tag{7}$$

To see (7), one may use the spectral theorem to write $(T^2x, x) = \int\limits_0^\infty t^2 \mathrm{d}\mu_x(t)$ for some measure $\mathrm{d}\mu_x$. Now by Jensen's inequality, we have

$$\|T^n x\|^2 = (T^{2n}x, x) = \int\limits_0^\infty t^{2n}\mathrm{d}\mu_x(t) \geqslant \left(\int\limits_0^\infty t^2 \mathrm{d}\mu_x(t)\right)^n = (T^2x, x)^n = \|Tx\|^{2n}.$$

In view of (7), with $\varepsilon_i = \|(A-B)x_i - \lambda x_i\|$, we now have

$$\begin{aligned}\|A^{2n}(A-B)+(A-B)B^{2n}\| &\geqslant |\lambda|(\|Ax_i\|^{2n}+\|Bx_i\|^{2n}) - (\|A^{2n}\|+\|B^{2n}\|)\varepsilon_i \geqslant \\ &\geqslant 2^{1-2n}|\lambda|(\|Ax_i\|+\|Bx_i\|)^{2n} - (\|A^{2n}\|+\|B^{2n}\|)\varepsilon_i \geqslant \\ &\geqslant 2^{1-2n}|\lambda|\,\|Ax_i - Bx_i\|^{2n} - (\|A^{2n}\|+\|B^{2n}\|)\varepsilon_i \geqslant \\ &\geqslant 2^{1-2n}|\lambda|(|\lambda|-\varepsilon_i)^{2n} - (\|A^{2n}\|+\|B^{2n}\|)\varepsilon_i\end{aligned}$$

Letting $i \to \infty$, we get $\|A^{2n}(A-B)+(A-B)B^{2n}\| \geqslant 2^{1-2n}\|A-B\|^{2n+1}$. Hence, according to (6) we finally obtain $2^{2n}\|A^{2n+1}-B^{2n+1}\| \geqslant \|(A-B)^{2n+1}\|$, as desired.

To treat the general case of unitarily invariant norms, we assume that $A^{2n+1} - B^{2n+1}$ belongs to $\mathcal{J}_{\|\|\cdot\|\|}$ and so $A^{2n+1}-B^{2n+1}$ is compact. By applying the usual operator norm case of (5) to the Calkin algebra setting we see that $A-B$ is also compact. Since $A-B$ is a compact self-adjoint operator, it follows that

$$(8) \qquad (A-B)e_j = \lambda_j e_j \quad \text{for } j = 1, 2, \ldots,$$

where $\{e_j\}$ is an orthonormal basis for H and $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$ is the sequence of singular values of $A-B$. Once again to prove (5) for all unitarily invariant norms, it is sufficient to prove it for the special class of Ky Fan norms. As in the usual operator norm case we have

$$(9) \qquad 2\|A^{2n+1}-B^{2n+1}\|_k \geqslant \|A^{2n}(A-B)+(A-B)B^{2n}\|_k.$$

Using a minimax principal of Ky Fan [6, p. 47], (7), (8) and (9) now imply

$$\begin{aligned}2\|A^{2n+1}-B^{2n+1}\|_k &\geqslant \sum_{j=1}^{k} |((A^{2n}(A-B)+(A-B)B^{2n})e_j, e_j)| = \\ &= \sum_{j=1}^{k} |\lambda_j|(\|A^n e_j\|^2 + \|B^n e_j\|^2) \geqslant \sum_{j=1}^{k} |\lambda_j|(\|Ae_j\|^{2n}+\|Be_j\|^{2n}) \geqslant \\ &\geqslant 2^{1-2n}\sum_{j=1}^{k} |\lambda_j|(\|(A-B)e_j\|^{2n} = 2^{1-2n}\sum_{j=1}^{k}|\lambda_j|^{2n+1} = 2^{1-2n}\|(A-B)^{2n+1}\|_k.\end{aligned}$$

Consequently, we have $2^{2n}\|A^{2n+1}-B^{2n+1}\|_k \geqslant \|(A-B)^{2n+1}\|_k$, and the proof of the theorem is now complete. ■

We conclude the paper with some corollaries. Our first corollary is an important special case of (5).

COROLLARY 1. *Let A, B be self-adjoint operators in $B(H)$. Then for every integer $n \geqslant 1$ and for every p with $1 \leqslant p \leqslant \infty$, we have*

$$(10) \qquad \|A-B\|^{2n+1}_{(2n+1)p} \leqslant 2^{2n}\|A^{2n+1}-B^{2n+1}\|_p.$$

Proof. Note that (10) follows from (5) specialized to the Schatten p-norms together with the fact that if T is a self-adjoint operator and m is any integer with $m \geqslant 1$, then $\|T^m\|_p = \|T\|_{mp}^m$ for $1 \leqslant p \leqslant \infty$. ■

It should be noted that (5) can be extended to nonself-adjoint operators by a familiar device of considering 2×2 operator matrices and direct sums. If the norm ideal $\mathcal{J}_{|||\cdot|||}$ is normed by the symmetric gauge function Φ, then so is $\mathcal{J}_{|||\cdot|||} \oplus \mathcal{J}_{|||\cdot|||}$ by the following procedure. Given two operators A and B in $\mathcal{J}_{|||\cdot|||}$, we define

$$|||A \oplus B||| = \Phi(\{s_1(A), s_1(B), s_2(A), s_2(B), \ldots\}),$$

which represents the $|||\cdot|||$-norm of $A \oplus B$ regarded as the operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $H \oplus H$ (see [3] and references therein). Note that $|||A \oplus A^*||| = |||A \oplus A|||$ and in particular we have

$$\|A \oplus A\| = \|A\| \text{ and } \|A \oplus A\|_p = 2^{1/p}\|A\|_p, \quad \text{for } 1 \leqslant p < \infty.$$

Another useful fact that is need in the proof of our second corollary says that $|||A \oplus A||| \leqslant |||B \oplus B|||$ for all unitarily invariant norms if and only if $|||A||| \leqslant |||B|||$ for all such norms. This is an immediate consequence of the Ky Fan dominance property.

COROLLARY 2. *Let A, B be operators in $B(H)$. Then for every integer $n \geqslant 1$ and for every unitarily invariant norm we have*

$$(11) \qquad |||\,|A - B|^{2n+1}||| \leqslant 2^{2n}|||A|A|^{2n} - B|B|^{2n}|||.$$

Proof. Consider the self-adjoint operators T, S defined on $H \oplus H$ by $T = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. Then it can be easily seen that

$$(T - S)^{2n+1} = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \text{ and } T^{2n+1} - S^{2n+1} = \begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix},$$

where $C = (A - B)|A - B|^{2n}$ and $D = A|A|^{2n} - B|B|^{2n}$. Apllying (5) to the operators T and S we obtain

$$|||C \oplus C^*||| \leqslant 2^{2n}|||D \oplus D^*|||, \text{ and, consequently } |||C||| \leqslant 2^{2n}|||D|||.$$

The desired inequality (11) now follows from this together with the fact that if X and Y are operators in $B(H)$, then for every unitarily invariant norm we have $|||XY||| = |||\,|X|Y|||$. ■

Our final corollary is related to the result of Macaev concerning Volterra operators which asserts that if A is a quasinilpotent operator in $B(H)$ (i.e. the spectrum of A

is $\{0\}$), such that $A - A^*$ belongs to $\mathcal{J}_p$ for some p with $1 < p < \infty$, then A itself belongs to $\mathcal{J}_p$, and moreover we have

$$\|A\|_p \leqslant c_p \|A - A^*\|_p, \tag{12}$$

where c_p is a constant depending only upon p (see [6, p. 215]).

It should be also noted that if A is a quasinilpotent operator such that $A - A^*$ is compact, then A must be compact. To see this, one needs to formulate the problem in the Calkin algebra setting (see also [12, p. 60]).

It has been recently shown in [8] that if A is a nilpotent operator (i.e. $A^n = 0$ for some integer $n > 1$) such that $A^*A - AA^*$ belongs to $\mathcal{J}_p$ for some p with $1 \leqslant p \leqslant \infty$, then A belongs to $\mathcal{J}_{2p}$ (in fact for $p = \infty$, the assertion is true under the weaker assumption that A is quasinilpotent). Also it is still possible in this case to have an estimate of the form

$$\|A\|_{2p}^2 \leqslant c_{p,n} \|A^*A - AA^*\|_p \quad \text{for } 1 \leqslant p \leqslant \infty \tag{13}$$

where $c_{p,n}$ is a constant depending on p and n.

In the same spirit we have the following related result.

COROLLARY 3. *Let A be a quasinilpotent operator in $B(H)$ such that $A|A|^{2n} - A^*|A^*|^{2n}$ belongs to $\mathcal{J}_p$ for some integer $n \geqslant 1$ and some p, with $1 < p \leqslant \infty$. Then A belongs to $\mathcal{J}_{(2n+1)p}$, and moreover we have*

$$\|A\|_{(2n+1)p}^{2n+1} \leqslant k_{p,n} \|A|A|^{2n} - A^*|A^*|^{2n}\|_p \quad \text{for } 1 < p \leqslant \infty, \tag{14}$$

where $k_{p,n}$ is a constant depending on p and n.

Proof. Applying Corollary 2 to A and $B = A^*$ with the norm $\|\cdot\|_p$, we see that $A - A^*$ belongs to $\mathcal{J}_{(2n+1)p}$, and hence A belongs to $\mathcal{J}_{(2n+1)p}$ by Macaev's result. Since $\|A - A^*\|_{(2n+1)p}^{2n+1} \leqslant 2^{2n} \|A|A|^{2n} - A^*|A^*|^{2n}\|_p$ for $1 \leqslant p \leqslant \infty$, the desired inequality (14) now follows from (12). ∎

REFERENCES

1. ANDO, T., Comparison of norms $|||f(A)-f(B)|||$ and $|||f(|A-B|)|||$, *Math. Z.*, **197**(1988), 403–409.
2. BHATIA, R.; KITTANEH, F., On the singular values of a product of operators, *SIAM J. Matrix Anal. Appl.*, **11**(1990), 272–277.
3. BHATIA, R.; KITTANEH, F., Norm inequalities for partitioned operators and an application, *Math. Ann.*, **287**(1990), 719–726.

4. BHATIA, R.; DAVIS, Ch., More matrix forms of the arithmetic-geometric mean inequality, *SIAM J. Matrix Anal. Appl.*, to appear.
5. BIRMAN, M. S.; KOPLIENKO, L. S.; SOLOMYAK, M. Z., Estimates for the spectrum of the difference between fractional powers of two self-adjoint operators (Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.*, **19**(1975), 3–10.
6. GOHBERG, I. C.; KREIN, M. G., *Introduction to the theory of linear nonselfadjoint operators*, American Mathematical Society, Providence, R.I., 1969.
7. KITTANEH, F., A note on the aritmetic-geometric mean ineaquality for matrices, *Linear Algebra Appl.*, to appear.
8. KITTANEH, F., Some trace class commutators of trace zero, *Proc. Amer. Math. Soc.*, to appear.
9. KOPLIENKO, L. S., On the theory of the spectral shift function, Topics in Mathematical Physics, Vol. 5, Consultants Bureau, New York, 1972.
10. MCINTOSH., A., Heinz inequalities and perturbation of spectral families, Macquarie mathematics Reports, Macquarie University, 1979.
11. NABOKO, S. N., Estimates in symmetrically normed ideals for the difference of powers of accretive operators (Russian), *Vestnik Leningrad Univ. Mat. Mekh. Astronom.*, **131**(1988), 40–45.
12. RINGROSE, J., *Compact non-self-adjoint operators*, Van Nostrand Reinhold, New York, 1971.
13. SCHATTEN, R., *Norm ideals of completely continuous operators*, Springer, Berlin, 1960.
14. SIMON, B., *Trace ideals and their applications*, Cambridge University Press, 1979.

DANKO JOCIĆ
Faculty of Mathematics,
Belgrade University,
Studentski trg 16A, 11000 Belgrade,
Yugoslavia

and

FUAD KITTANEH
Department of Mathematics,
Indiana University,
Bloomington, IN 47405,
U.S.A.

Received April 1, 1991; revised June 17, 1991.