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A HILBERT-SCHMIDT NORM EQUALITY ASSOCIATED WITH THE FUGLEDE-PUTNAM-ROSENBLUM'S TYPE THEOREM FOR GENERALIZED MULTIPLIERS

DANKO R. JOCIĆ

1. INTRODUCTION

Let H denote a separable, infinite-dimensional, complex Hilbert space. Let $C_2(H) \subset B(H)$, $\|\cdot\|_2$, $\sigma(\cdot)$, $\lceil$ denote, respectively, the Hilbert-Schmidt class, the class of all bounded linear operators, the Hilbert-Schmidt norm, the spectrum and the restriction of an operator. In this paper, for normal operators A and B we will always denote by $E(\cdot)$ and $B(\cdot)$ respectively their associated spectral measures.

The following theorem:

THEOREM 1.1. *For A, B, $X \in B(H)$ with A and B normal, $\Delta(X) \equiv AX - XB = 0$ implies $\Delta^*(X) \equiv A^*X - XB^* = 0$,*

is known as the classical Fuglede-Putnam-Rosenblum's (FPR) theorem" (see [15]). This theorem has been generalized in various ways under different conditions for A, B and X, and for different (mainly differential) expresions Δ (and appropriately defined) Δ^* (see [16], [2], [3], [26]). The aymptotic versions of those generalizations have also been proven in different operator topologies by use of R.L. Moore's construction (see [18], [20]).

On the other hand, the commutator $\Delta(A, B)X = AX - XB$ and the multiplier $M(A, B)X = AXB$ are the simplest cases of generalized multipliers, associated to analytic functions $f(z, w) = z - w$ and $f(z, w) = zw$ respectively. Various aspects of generalized multipliers are investigated, for example, in [9], [25], [11], [4], [5], [6], [7].

In this paper we prove an FPR type theorem for generalized multipliers.

2. ESSENTIALLY BOUNDED FUNCTIONAL CALCULUS

THEOREM 2.1. (Boundedness of the trace variation) *For arbitrary* $X, Y \in C_2(H)$ *we have*

$$\sum_{m,n} |\mathrm{tr}(E(\gamma_m)XF(\delta_n)Y^*)| \leqslant \|X\|_2\|Y\|_2,$$

for every finite Borel partition $\{\gamma_m\}$, $\{\delta_n\}$ *of the complex plane.*

Proof.

$$\sum_{m,n} |\mathrm{tr}(E(\gamma_m)XF(\delta_n)Y^*)| =$$

$$= \sum_{m,n} |\mathrm{tr}(E(\gamma_m)XF(\delta_n)(E(\gamma_m)YF(\delta_n))^*)| \leqslant$$

$$\leqslant \sum_{m,n} |\mathrm{tr}(E(\gamma_m)XF(\delta_n)(E(\gamma_m)XF(\delta_n))^*)\mathrm{tr}(E(\gamma_m)YF(\delta_n)(E(\gamma_m)YF(\delta_n))^*)|^{1/2} \leqslant$$

$$\leqslant \left\{\sum_{m,n} \mathrm{tr}(E(\gamma_m)XF(\delta_n)(E(\gamma_m)XF(\delta_n))^*)\right\}^{1/2} \times$$

$$\times \left\{\sum_{m,n} \mathrm{tr}(E(\gamma_m)YF(\delta_n)(E(\gamma_m)YF(\delta_n))^*)\right\}^{1/2} =$$

$$= \left\{\sum_{m,n} \mathrm{tr}(E(\gamma_m)XF(\delta_n)X^*)\right\}^{1/2} \left\{\sum_{m,n} \mathrm{tr}(E(\gamma_m)YF(\delta_n)Y^*)\right\}^{1/2} =$$

$$= \mathrm{tr}(XX^*)^{1/2}\mathrm{tr}(YY^*)^{1/2} = \|X\|_2\|Y\|_2.$$ ■

According to the extension theorem given in [23] we derive the following:

COROLLARY 2.1. *Let* $X, Y \in C_2(H)$. *The mapping* $\gamma \times \delta \to \mathrm{tr}(E(\gamma)XF(\delta)Y^*)$ *of the family of all Cartesian products of Borel sets in* $\mathbb{C}$ *can be extended in a unique way to a complex Borel measure* $\mu_{X,Y}$ *in* $\mathbb{C}^2$ *such that* $|\mu_{X,Y}| \leqslant \|X\|_2\|Y\|_2$.
Also, the mapping $f \to \int_{\mathrm{supp}E\times\mathrm{supp}F} f d\mu_{X,Y}$ is a bounded linear functional on $L_\infty(\mathrm{supp}E\times\mathrm{supp}F,\, d\mu_{X,Y})$, because

$$\left|\int_{\sigma(A)\times\sigma(B)} f d\mu_{X,Y}\right| \leqslant \|f\|_\infty|\mu_{X,Y}| \leqslant \|f\|_\infty\|X\|_2\|Y\|_2.$$

Sometimes, we use some more informative notations

$$\int_{\mathrm{supp}E\times\mathrm{supp}F} f(z,w)\mathrm{d}\,\mathrm{tr}(E(z)XF(w)Y^*)$$

and

$$\int_{\operatorname{supp}E\times\operatorname{supp}F} f(z,w)\operatorname{tr}(\mathrm{d}E(z)X\,\mathrm{d}F(w)Y^*)$$

instead of

$$\int_{\operatorname{supp}E\times\operatorname{supp}F} f\,\mathrm{d}\mu_{X,Y}.$$

From the same inequality we conclude that for every $Y \in C_2(H)$ and every function f, which is bounded and Borel measurable on $\operatorname{supp}E \times \operatorname{supp}F$, the mapping $X \to \to \int_{\operatorname{supp}E\times\operatorname{supp}F} f\,\mathrm{d}\mu_{X,Y}$ is a bounded linear functional on $C_2(H)$, which is a Hilbert space itself (see [17]), with the norm given by $\langle A, B\rangle_2 = \operatorname{tr}(AB^*)$. Therefore there exists a bounded linear operator on $C_2(H)$, denoted by $f(A,B)$ or $\Delta_f(A,B)$, such that

$$\operatorname{tr}(f(A,B)(X)Y^*) = \int_{\operatorname{supp}E\times\operatorname{supp}F} f(z,w)\mathrm{d}\operatorname{tr}(E(z)XF(w)Y^*).$$

Obviously, $\|f(A,B)\| \leqslant \|f\|_\infty$, but some more precise considerations show that this inequality is in fact an equality.

Some of the well known functional calculus formulae for normal operators (see [8], p.131, (4)–(11) and p.154 (10)) can be rephrased as follows:

(i) $(\alpha f + \beta g)(A,B)X = \alpha f(A,B)X + \beta g(A,B)X$,

(ii) $(fg)(A,B)X = f(A,B)(g(A,B)X)$,

(iii) $\sigma(f(A,B))$ is exactly the essential range of $\sigma(A)\times\sigma(B)$ by f, being measured by $\mu\times\nu$, for any positive measures μ and ν mutualy absolutely continuous with respect to E and F respectively.

(iv) $f(A,B)^* = \overline{f}(A,B)$.

(v) $f(A,B)X = AX$ if $f(z,w) = z$ and $f(A,B)X = XB$ if $f(z,w) = w$.

From (ii) and (iv) it follows that $f(A,B)$ is a normal operator on $C_2(H)$, and therefore, if $X \in C_2(H)$, the Hilbert-Schmidt equality in FPR type theorem for (essentially) bounded multipliers coincides, in fact, with the normality of the corresponding multiplier. But if X is in $B(H)$ but not in $C_2(H)$, this question has a sense only if $f(A,B)X$ is definable and belongs to $C_2(H)$. One posible situation is when f is analytic, and so, we will concentrate our attention on this type of multipliers, to which we will, in the sequel, refer as to analytic or generalized multipliers.

3. GENERALIZED MULTIPLIERS

Following [11], for $A, B \in B(H)$, and for every function f analytic in each variable in some neighbourhood of the set $\sigma(A)\times\sigma(B)$, we can define on $B(H)$ a linear operator

$f(A,B)$, which is called a generalized (or analytic) multiplier (or transformator), by

$$f(A,B)(X) = -\frac{1}{4\pi^2}\oint_{\Gamma_A}\oint_{\Gamma_B} f(z,w)(A-z)^{-1}X(B-w)^{-1}\mathrm{d}z\ \mathrm{d}w,$$

for every operator $X \in B(H)$. The above integrals are calculated over regular contoures Γ_A and Γ_B surrounding $\sigma(A)$ and $\sigma(B)$ respectively. If the operators A and B are known from context, we also use the notation Δ_f, and if in addition the function f is known, we denote it simply by Δ. The last notation is traditional and has come from the broadly investigated transformator $X \to AX - XB$ (see for example [1], [12], [13], [14], [21], [11]).

Similarly, we define the adjoint multiplier:

$$f(A,B)^*(X) = -\frac{1}{4\pi^2}\oint_{\overline{\Gamma}_A}\oint_{\overline{\Gamma}_B} \overline{f(\overline{z},\overline{w})}(A^*-z)^{-1}X(B^*-w)^{-1}\mathrm{d}z\ \mathrm{d}w$$

(where $\overline{\Gamma}_A$ stands for $\{z\colon \overline{z} \in \Gamma_A\}$).

For such functional calculus (see [19], [11], [24]) the formulae (i), (ii) and (v) are still valid, while (iii) and (iv) become

(iii) $\sigma(f(A,B)) \subset f(\sigma(A),\sigma(B))$,

(iv′) $\big(f(A,B)\lceil_{C_2(H)}\big)^* = f(A,B)^*\lceil_{C_2(H)}$.

Of course, for normal operators A and B and such an analytic f, a straightforward application of the Cauchy reproducing formula shows that this transformator coincide with the previously introduced one.

4. FPR THEOREM

In the beginning we give some well-known theorems concerning the factorization of an analytic function.

THEOREM 4.1. (Weierstrass, see [10, p.11]) *Let f be an analytic (in each variable) function in the open set $W' = U \times \{w\colon |w| \leqslant R\}$, where U is a neighbourhood of the origin 0 in $\mathbb{C}$ such that $f(0,w) \not\equiv 0$ in the disc $\{w\colon |w| \leqslant R\}$. Let also $r \leqslant R$ for some r such that $f(0,w)$ has no zero on the sphere $\{w\colon |w| = r\}$ and let k be the number of zeros of the same function in the open disc $V_r = \{w\colon |w| \leqslant r\}$ counting their multiplicity. Then there exists a neighbourhood $W' = U' \times V_r \subset W$ of the origin in $\mathbb{C}^2$ in which the function f can be represented in the following form:*

$$f(z,w) = \big(w^k + c_1(z)w^{k-1} + \cdots + c_k(z)\big)\, f_0(z,w),$$

for some functions $c_j(z)$ *in* U', *and for some analytic function* f_0 *having no zeros in* W'.

THEOREM 4.2. (Factorization theorem, see [10, p.13]) *Let* U *be a (simply connected) open set in* $\mathbb{C}$ *and let the function* f *be analytic in* $U \times V$ *such that for every fixed* $z \in U$ *the function* $f(z,\cdot)$ *has in* V *exactly* m *geometricaly different zeros. Then those zeros analyticly depend on* z, *i.e. there exist: analytic in* U *functions* $\{\alpha_i\}_{i=1}^m$, *a function* f_0, *analytic and with no zeros in* $U \times V$, *and some natural numbers* $\{k_i\}_{i=1}^m$ *such that*

$$f(z,w) = \prod_{i=1}^{m}(w - \alpha_i(z))^{k_i} f_0(z,w)$$

for every $(z,w) \in U \times V$.

THEOREM 4.3. (Discriminant set, see [10, p.11]) *Let the function* f *be analytic in the bounded open set* $W = U \times V$ *in* $\mathbb{C}^2$ *with zero set having no accumulation points on* $\partial U \times V$. *If* $m < \infty$ *is the maximal number of geometricaly different zeros of* $f(z,w)$ *in* V *with* $z \in U$ *fixed, then* G, *the set of points at which this maximum is obtained, is an open set everywhere dense in* U. *Moreover, there is a function* $\Delta(z)$ *analytic in* U *(not identical to zero) having a set* $U \setminus G$ *for its zero set (on* U*).*

COROLLARY 4.1. (see [10, p.15]) *Let the function* f *be analytic in the bounded open set* $W = U \times V$ *in* $\mathbb{C}^2$ *with zero set having no accumulation points on* $U \times \partial V$. *Then there is a Weierstrass polynomial* F *such that* $Z_F = Z_f$ *and such that for every fixed* $z \in G$ *all the roots of the polynomial* $F(z,\cdot)$ *are simple.*

COROLLARY 4.2. (see [10, p.15]) *The discriminant set of* f, *i.e. the projection on* U *of the set*

$$\left\{(\xi,\eta) \in U \times V : F(\xi,\eta) = \frac{\partial}{\partial\eta}F(\xi,\eta) = 0\right\},$$

is the zero set of the above mentioned polynomial F.

Now we give our main FPR theorem for generalized multipliers.

THEOREM 4.4. (FPR theorem for generalized multipliers) *Let* A, B, $X \in B(H)$, *with* A *and* B *normal, and let* f *be analytic in some neighbourhood of the set* $\sigma(A)\times$ $\times\sigma(B)$. *If* $\Delta_f(X) \in C_2(H)$, *then* $\Delta_f^*(X) \in C_2(H)$ *with*

$$\|\Delta_f(X)\|_2 = \|\Delta_f^*(X)\|_2.$$

Proof. Using the additivity of the spectral measure, it is sufficient to prove that for every connected component $U \times V$ of the above mentioned neighbourhood of the set $\sigma(A) \times \sigma(B)$, we have $\Delta_f^*(E(U)XF(V)) \in C_2(H)$ and

$$\|\Delta_f(E(U)XF(V))\|_2 = \|\Delta_f^*(E(U)XF(V))\|_2.$$

If $f \equiv 0$ this is obvious.

If $f \not\equiv 0$, then the set $\sigma_0 := \{z \in \sigma(A): (\forall w \in V) f(z,w) = 0\}$ is finite according to the theorem of uniqueness. For every $\varepsilon > 0$ let $\sigma_\varepsilon := \sigma(A) \setminus \bigcup_{s \in \sigma_0} B(s,\varepsilon)$ and $U_\varepsilon := := U \setminus \bigcup_{s\in\sigma_0} \overline{B}(s,\varepsilon/2)$. Obviously $\sigma_\varepsilon \subset U_\varepsilon$. Also, for the operator $A_\varepsilon := A\lceil_{E(\sigma_\varepsilon)H}$, we have $\sigma(A_\varepsilon) = \sigma_\varepsilon$.

For every $(z,w) \in \sigma(A_\varepsilon) \times \sigma(B)$ there exists $r_{z,w} > 0$ such that $V_{z,w} := := B(w, r_{z,w}) \subset V$ and such that the function $f(z,\cdot)$ has no zeros on $\partial V_{z,w}$ (if $f(z,w) = 0$ it exists because zero w is isolated, and if $f(z,w) \neq 0$, by continuity). By continuity of f and compactness of $\partial V_{z,w}$ there is $\varepsilon_{z,w} > 0$ and there is $U_{z,w} := B(w, 2\varepsilon_{z,w}) \subset U_\varepsilon$, such that $f \neq 0$ on $U_{z,w} \times \partial V_{z,w}$.

So we have that m, the maximal number of geometricaly different zeros of $f(\cdot,w)$ on $U_{z,w}$, is finite, and also, that $D_{z,w}$, the set of points where this maximal number is **not** obtained is exactly the set

$$\left\{\xi \in U_{z,w}: (\exists \eta \in V_{z,w}) F(\xi,\eta) = \frac{\partial}{\partial \eta} F(\xi,\eta) = 0\right\},$$

where F is the Weierstrass polynomial associated to f on $U_{z,w}$. It also coincide with the set of zeros of the discriminant of F on $U_{z,w}$, which is an analytic function, and thus we have $\sigma_{z,w} := D_{z,w} \cap \overline{B}(w, \varepsilon_{z,w})$ to be finite. So, for every $\varepsilon > 0$, the set $\sigma_1 := \bigcup_{\xi \in \sigma_\varepsilon} D_{z,w}$ is finite according to compactness of σ_ε. Once again, that will allow us to "eliminate" a small neighbourhouds of σ_1 by letting $\sigma_{\varepsilon'} := \sigma(A_\varepsilon) \setminus \bigcup_{s\in\sigma_1} B(s, 2\varepsilon')$ and $U_{\varepsilon'} := U_\varepsilon \setminus \bigcup_{s \in \sigma_1} B(s,\varepsilon')$ for an arbitrary $\varepsilon' > 0$. Obviously $\sigma_{\varepsilon'} \subset U_{\varepsilon'}$, and, for a given $(z,w) \in \sigma_{\varepsilon'} \times \sigma(B)$ there is $\varepsilon'_{z,w} > 0$ such that $U'_{z,w} := B(z, \varepsilon'_{z,w}) \subset U_{z,w} \cap U_{\varepsilon'}$. Since the zero set of f has no accumulation points on $U'_{z,w} \times \partial V_{z,w}$, and since the number of geometricaly different zeros of $f(z,\cdot)$ is constant, we will have, according to Theorem 4.2 that

$$f(\xi,\eta) = \prod_{k=0}^{n} f_k(\xi,\eta)$$

on $U'_{z,w} \times V_{z,w}$, for some $n \in \mathbf{N}$, with $f_k(\xi,\eta) = \eta - a_k(\xi)$ for some analytic (in $V_{z,w}$) functions $a_k(\xi)$, for every $1 \leqslant k \leqslant n$ and some $f_0(\xi,\eta)$ analytic and having no zeros in $U'_{z,w} \times V_{z,w}$.

Define

$$A_{z,w} = A\lceil_{E(U'_{z,w})H},$$

$$B_{z,w} = B\lceil_{F(V_{z,w})H}$$

and

$$X_{z,w} = E\left(U'_{z,w}\right) X A\lceil_{F(V_{z,w})H}.$$

The commutativity of the operator family $\{\Delta_{f_k}, \Delta^*_{f_k}\}_{k=0}^n$ on $B(E(U'_{z,w})H, F(V_{z,w})H)$ and Theorem 1 in [26] consequently give

$$\left\|\Delta_{f_1}\left(\prod_{k=2}^n \Delta_{f_k}(\Delta_{f_0}(X_{z,w}))\right)\right\|_2 = \left\|\Delta^*_{f_1}\left(\prod_{k=2}^n \Delta_{f_k}(\Delta_{f_0}(X_{z,w}))\right)\right\|_2$$

$$= \left\|\prod_{k=2}^n \Delta_{f_k}\left(\Delta^*_{f_1}(\Delta_{f_0}(X_{z,w}))\right)\right\|_2 .$$

Using the same arguments $n-1$ times we get

$$\left\|\Delta_{f_0}\left(\prod_{k=1}^n \Delta_{f_k}(X_{z,w})\right)\right\|_2 = \left\|\prod_{k=1}^n \Delta^*_{f_k}(\Delta_{f_0}(X_{z,w}))\right\|_2 = \left\|\Delta_{f_0}\left(\prod_{k=1}^n \Delta^*_{f_k}(X_{z,w})\right)\right\|_2 .$$

Since the transformator $\Delta^*_{f_0}\Delta^{-1}_{f_0}$ is an isometry on $B(E(U'_{z,w})H, F(V_{z,w})H)$, then the above expressions are equal to

$$\left\|\Delta^*_{f_0}\left(\prod_{k=1}^n \Delta_{f_k}(X_{z,w})\right)\right\|_2 ,$$

and this implies

$$\|\Delta_f(X_{z,w})\|_2 = \|\Delta^*_f(X_{z,w})\|_2,$$

and hence

$$\|E(U'_{z,w})\Delta_f(X)F(V_{z,w})\|_2 = \|E(U'_{z,w})\Delta^*_f(X)F(V_{z,w})\|_2.$$

According to the compactness of $\sigma(A_{\varepsilon'})\times\sigma(B)$, there is a finite covering $\{U'_{z_i,w_i}\times \times V_{z_i,w_i}\}_{i=1}^I$, which, according to the additivity of vector functions μ, μ^* defined by

$$\mu(\gamma\times\delta) = E(\gamma)\Delta_f(X)F(\delta)$$

and

$$\mu^*(\gamma\times\delta) = E(\gamma)\Delta^*_f(X)F(\delta),$$

gives

$$\left\|\mu^*\left(\bigcup_{i=1}^I U'_{z_i,w_i}\times V_{z_i,w_i}\right)\right\|_2 = \left\|\mu\left(\bigcup_{i=1}^I U'_{z_i,w_i}\times V_{z_i,w_i}\right)\right\|_2 . \tag{1}$$

Still more, the strong continuity of every spectral measure in the corresponding Hilbert space gives that the measure G_Z, defined by

$$G_Z(\gamma\times\delta) = E(\gamma)ZF(\delta),$$

is additive and strongly continuous for all $Z \in B(H)$ and σ-additive and C_2-continuous whenever $Z \in C_2(H)$.

Since σ_1 is finite, the strong continuity of μ^* and C_2-continuity of μ gives

$$s - \lim_{\varepsilon' \to 0} E\left(\bigcup_{s \in \sigma_1} B(s, 2\varepsilon') \setminus \sigma_1\right) \Delta_f^*(X) = 0$$

and

$$C_2 - \lim_{\varepsilon' \to 0} E\left(\bigcup_{s \in \sigma_1} B(s, 2\varepsilon') \setminus \sigma_1\right) \Delta_f(X) = 0.$$

Having

$$(\sigma_e \setminus \sigma_1) \times \sigma(B) \setminus \bigcup_{i=1}^{I} (U_{z_i,w_i} \times V_{z_i,w_i}) \subset \left(\bigcup_{s \in \sigma_1} B(s, 2\varepsilon') \setminus \sigma_1\right) \times \sigma(B),$$

we obtain

$$s - \lim_{\varepsilon' \to 0} \mu^* \left(\bigcup_{i=1}^{I} U'_{z_i,w_i} \times V_{z_i,w_i}\right) = E(\sigma_e \setminus \sigma_1)\Delta_f^*(X)$$

and

$$C_2 - \lim_{\varepsilon' \to 0} \mu \left(\bigcup_{i=1}^{I} U'_{z_i,w_i} \times V_{z_i,w_i}\right) = E(\sigma_e \setminus \sigma_1)\Delta_f(X).$$

Now, having both sides of equality (1) less or equal to $\|\Delta_f(X)\|_2$, the uniform boundedness principle on $C_2(H)$ gives $E(\sigma_e \setminus \sigma_1)\Delta_f^*(X) \in C_2(H)$, but this imply that we also have

$$C_2 - \lim_{\varepsilon' \to 0} \mu^* \left(\bigcup_{i=1}^{I} U'_{z_i,w_i} \times V_{z_i,w_i}\right) = E(\sigma_e \setminus \sigma_1)\Delta_f^*(X).$$

The limit process in (1), together with $\|X\overline{f}(s, B)\|_2 = \|Xf(s, B)\|_2$ for all $s \in \sigma_1$ gives

$$\|E(\sigma_e)\Delta_f^*(X)\|_2 = \|E(\sigma_e)\Delta_f(X)\|_2.$$

The similar procedure with ε instead of ε', talking account that $X\overline{f}(\overline{s}, B^*) =$ $= Xf(s, B) = 0$ for all $s \in \sigma_0$, gives $\Delta_f^*(X) \in C_2(H)$, and finally

$$\|\Delta_f^*(X)\|_2 = \|\Delta_f(X)\|_2. \qquad \blacksquare$$

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DANKO R. JOCIĆ
Institute for Applied Mathematics
and Electronics,
Belgrade,
Yugoslavia.

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